On the Moduli of a Quantized Elastica in $\mathbb{P}$ and KdV Flows: Study of Hyperelliptic Curves as an Extension of Euler's Perspective of Elastica I

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Abstract

Quantization needs evaluation of all of states of a quantized object rather than its stationary states with respect to its energy. In this paper, we have investigated moduli $\mathcal{M}_{\mathrm{elas}}^{\mathbb{P}}$ of a quantized elastica, a quantized loop with an energy functional associated with the Schwarz derivative, on a Riemann sphere $\mathbb{P}$. Then it is proved that its moduli space is decomposed to a set of equivalent classes determined by flows obeying the Korteweg-de Vries (KdV) hierarchy which conserve the energy. Since the flow obeying the KdV hierarchy has a natural topology, it induces topology in the moduli space $\mathcal{M}_{\mathrm{elas}}^{\mathbb{P}}$. Using the topology, $\mathcal{M}_{\mathrm{elas}}^{\mathbb{P}}$ is classified.

Studies on a loop space in the category of topological spaces **Top** are well-established and its cohomological properties are well-known. As the moduli space of a quantized elastica can be regarded as a loop space in the category of differential geometry **DGeom**, we also proved an existence of a functor between a triangle category related to a loop space in **Top** and that in **DGeom** using the induced topology.

As Euler investigated the elliptic integrals and its moduli by observing a shape of classical elastica on $\mathbb{C}$, this paper devotes relations between hyperelliptic curves and a quantized elastica on $\mathbb{P}$ as an extension of Euler's perspective of elastica.

§1. Introduction

History of investigations of elastica was opened by James Bernoulli in 1691 according to Truesdell's inquiry [T1, 2, L]. He named a shape of a thin non-stretching elastic rod elastica and proposed the elastica problem: what shape does elastica take for a given boundary condition? It should be, further, noted that he also proposed the lemniscate problem and discovered an elliptic integral corresponding to the lemniscate function by investigation of elastica. He considered a smooth curve with the arc-length in a plane $\mathbb{C}$,

$$\tilde{\gamma} : [0, l] \hookrightarrow \mathbb{C}, \quad (s \mapsto \tilde{\gamma}(s)).$$

Following his studies, his nephew Daniel Bernoulli discovered that the elastica obeys the minimal principle that shape of the elastica is realized as a stationary point of an energy functional, which is called Euler-Bernoulli functional nowadays,

$$E[\tilde{\gamma}] = \int k^2 ds,$$

where k is the curvature of the curve $\tilde\gamma$ in $\mathbb{C}$, $k = -\sqrt{-1}\partial_s^2\tilde\gamma/\partial_s\tilde\gamma$, $\partial_s := d/ds$, and s is the arc-length of the curve using the induced metric in $\mathbb{C}$. (It should be noted that this functional *differs* from that of a "string" in the literature of the string theory in the elementary particle physics: although an elastica is a model of a string of the chord *e.g*, the guitar, "string" in the string theory can not be realized in the classical mechanical regime.)

Since the curvature k is expressed as $k = \partial_s\phi$ where ϕ is the tangential angle, and the energy is given by $E = \int |\partial_s\phi|^2 ds$, the elastica problem could be interpreted as the oldest problem of a harmonic map into a target space U(1); if we write $\partial_s\phi ds = g^{-1}dg$, for U(1) valued function g over $\mathbb{C}$, then the Hodge-star dual $*g^{-1}dg = \partial_s\phi$ and $E = \int < g^{-1}dg \wedge *g^{-1}dg >$.

The elastica problem is to investigate moduli $\tilde{\mathcal{M}}_{\text{elas,cls}}$,

$$\tilde{\mathcal{M}}_{\text{elas,cls}} := \{\tilde\gamma : [0,1] \hookrightarrow \mathbb{C} \mid \delta E[\tilde\gamma]/\delta\tilde\gamma = 0\}/\sim .$$

Here "$\sim$" means modulo Euclidean move in $\mathbb{C}$ and dilatation. We sometimes call this space *moduli space of the classical elasticas*. The classification of this moduli space $\tilde{\mathcal{M}}_{\text{elas,cls}}$ was essentially done by Euler in 1744 by means of numerical computations [E]. The moduli space $\tilde{\mathcal{M}}_{\text{elas,cls}}$ is classified by the moduli of the elliptic curves [T1, 2, L, We]. It is noted that before Euler refereed to Fagnano's paper on his discovery of an algebraic properties of the lemniscate function (an elliptic function of a special modulus) at December 31 1751, the elliptic integrals for more general modulus was investigated in the study of this classical harmonic map problem. (It is known that Jacobi recognized that the day is the birthday of the elliptic function. Thus we think that elastica is a kind of the movements of the fetus of algebraic curves.) We also emphasize that from the beginning, the harmonic map problem (classical field theory in physics) is closely related to algebraic varieties. Recently Mumford investigated this elastica problem from a viewpoint of applied mathematics and gave simple and deep expressions of the shape of elastica, which show the depth, importance and beauty of this problem [Mum3].

Especially for a closed elastica, Euler showed that its moduli space,

$$\mathcal{M}_{\text{elas,cls}} := \{\gamma : S^1 \hookrightarrow \mathbb{C} | \delta E[\gamma]/\delta\gamma = 0\}/\sim,$$

consists of two disjoint points: the corresponding moduli τ of the elliptic curves consist of two points $\tau = 0$ and $\tau = 0.70946\cdots$ [E, T1, 2, L].

Recently a loop space is one of the most concerned objects in mathematics and there have been so many efforts to investigate it [Br, G, LP, Se, SW and reference therein]. Further it is well-known that soliton equations are closely related to the loop spaces, loop groups and loop algebras [G, SW]. However these studies are sometimes too abstract to be related to physical problems, except problems in the elementary particle physics; for example, the embedded space is often a group manifold, *e.g.*, U(N). Further the energy function is paid little attention in these studies.

On the other hand, our concerned object is a non-stretching elastica, which is related to a large polymer, such as the deoxyribonucleic acid (DNA) as a physical model [Mat1, 2, 4, KV]. Elastica has an energy functional as we described above. Thus our problem, basically, differs from the arguments in an ordinary loop space in [G, Se, SW] except [Br, LP] though it is closely related to them.

One of these authors (S.M.) considered the quantization of a closed elastica (precisely speaking, statistical mechanics of elasticas) [Mat2]. He defined the moduli space of the closed quantized elastica, which is an isometric immersion of S^1 into $\mathbb{C}$ module the Euclidean motion and dilatation,

$$\mathcal{M}^{\mathbb{C}}_{\text{elas}} := \{\gamma : S^1 \hookrightarrow \mathbb{C} | \text{ isometric immersion }\}/\sim .$$

He investigated the partition function from a physical point of view, which has not been mathematically justified:

$$Z : \mathcal{M}^{\mathbb{C}}_{\text{elas}} \times \mathbb{R}_{>0} \longrightarrow \mathbb{R},$$

with

$$Z[\beta] = \int_{\mathcal{M}^{\mathbb{C}}_{\text{elas}}} D\gamma \exp(-\beta E[\gamma]),$$

where $\beta \in \mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$ and $D\gamma$ is the Feynman measure. On the quantization of an elastica, we need more information of the moduli space of curves besides those around its stationary points. To evaluate this map Z, he classified the moduli space of a quantized closed elastica $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$ and attempted to redefine the Feynman measure by replacing it with the series of Riemann integral over $\mathcal{M}^{\mathbb{C}}_{\text{elas}}$. His quantization is somewhat novel for an elastica. He physically proved that the moduli space of the quantized elastica is given as a subspace of the moduli space of the modified Korteweg-de Vries (MKdV) equation [Mat2].

Here we should emphasize that it is very surprising that a physical system is completely described by a soliton equation as mentioned in [Mat2]. Even in physical phenomena which are known as systems represented by soliton equations, like shallow waves, plasma waves, charge density waves and so on, the higher soliton solutions are, in general, out of their approximation regions; of course one or two soliton solutions do represent these phenomena well. On the other hand, in the quantized elastica problem, its functional space is completely expressed by the MKdV hierarchy, even though problems in polymer physics are, in general, too complex to be solved exactly [KV].

In this paper, we will rewrite the physical theorem in [Mat2] from a mathematical point of view and extend it. Pedit gave a lecture on a loop space over a Riemann sphere $\mathbb{P}$ at Tokyo Metropolitan University in 1998 [Ped]. There he showed that the loop space is related to the Korteweg-de Vries (KdV) flow by considering a loop in $\mathbb{C}^2 \setminus \{0\}$. As his treatment is given in the framework of pure mathematics, we will follow the expressions of Pedit and deal with the KdV flow instead of the MKdV flow here. Due to the Miura map (the Ricatti type differential equation), the MKdV flow and the KdV flow can be regarded as different aspects of the same object; this choice is not significant. Mathematical investigations on the KdV flow leads us to our main results, Theorems 3-4, 4-2 and 7-4.

As we will show later, our investigation of a quantized elastica leads us to study the hyperelliptic curves and their moduli space as Euler encountered the elliptic integrals and studied of the moduli of the elliptic functions by observing a shape of classical elastica on $\mathbb{C}$. One of our purposes of this study is to know the hyperelliptic functions and its moduli by investigating a quantized elastica in $\mathbb{P}$ as an extension of Euler's perspective of elastica. After we submitted the first version of this paper, these works progressed [Mat7-10]. Hence in this revised version, we also rewrite the related parts.

Contents of this paper is as follows.

§2 shows an expression of a real curve immersed in a Riemann sphere $\mathbb{P}$ according to the lecture of Pedit [Ped]. Using his expressions, we define the moduli space of a real smooth curve immersed in $\mathbb{P}$ and an energy functional of the curve whose integrand is the Schwarz derivative along the curve. When we regard $\mathbb{P}$ as complex plane with the infinity point, $\mathbb{C} \cup \{\infty\}$, the energy functional is identified with the Euler-Bernoulli energy functional around the origin $\{0\}$ of $\mathbb{C} \cup \{\infty\}$ and the curve with the energy is reduced to a quantized elastica which was studied by one of these authors [Mat2]. Thus we continue to refer such a curve in $\mathbb{P}$ "quantized elastica in $\mathbb{P}$". In order to consider a quantum effect, we should get knowledge of a set of curves with different energies instead of investigation of only a stationary point of the energy functional even though we are dealing with a single elastica. Thus we will call, in this paper, the moduli $\mathbb{M}^{\mathbb{P}}_{\text{elas}}$ defined in Definition 2-10 and 2-12, "moduli of a quantized elastica" rather than moduli of loops. In §2, we will give an equivalence between a loop space over $\mathbb{C}^2 \setminus \{0\}$ and $\mathbb{P}$ in a certain sense. Further following MacLaughlin and Beylinski [McLau, Br], we will introduce a natural topology of the loop space which is induced from the topology of the base space.

In §3, we introduce infinite dimensional parameters $t = (t_1, t_2, t_3, \cdots)$ which deform a given curve and define a flow obeying the KdV hierarchy along t, which is called KdVH flow. First we give our first main Theorem 3-4 in this paper. Since the energy functional of a curve turns out to be the first integral with respect to the parameter t, we prove that using the KdVH flow we can classify the moduli $\mathbb{M}^{\mathbb{P}}_{\text{elas}}$ of a quantized elastica in $\mathbb{P}$. In other words, the moduli space $\mathbb{M}^{\mathbb{P}}_{\text{elas}}$ is decomposed to a

set of the equivalent classes with respect to the KdVH flow. As §3 gives the differential geometrical and dynamical properties of the quantized elastica, we will attempt to express the theorem in terms of the words of the differential geometers. Remark 3-10 is a key of the study in §3.

Primary considerations leads the fact that the moduli space of a quantized elastica $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ is a subspace of the moduli space of the KdVH flow $\mathbb{M}_{\rm KdV}$ as shown in Proposition 4-29. The system of the KdV hierarchy has a natural topology, which essentially determines the algebraic properties of the KdV hierarchy [D, S, SN, SS, SW]. Using results on these studies of the KdV hierarchy, we give finer classification of the moduli space $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ in Theorem 4-2 and Proposition 4-33, which is our second main theorem. There a dense subspace in $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ is decomposed by a subspace characterized by a natural number. As we defined below Lemma 4-1, we encounter a finite type of the KdVH flow, which corresponds to the finite type solutions of the KdV equation and are related to a hyperelliptic curve. The natural number is related to genus of the hyperelliptic curve.

In order that we mention our second main statements, Theorem 4-2 and Proposition 4-33, §4 reviews the algebro-geometrical properties of the KdV hierarchy based upon the so-called Sato-Mulase theory [Mul, SS, SN]. As the completion of set of finite type solutions is equal to $\mathcal{M}_{\rm KdV}$, we concentrate our attention on the finite solution of the KdV flow and consider $\mathcal{M}_{\rm KdV}$ algebro-geometrically. As Sato-Mulase theory is of the algebraic analysis and is based upon the formal power series ring, we replace the base ring of smooth functions by the formal power series. There we find that a commutative differential ring is connected with geometry of a commutative ring, *i.e.*, a hyperelliptic curve. Using the inclusion $\mathcal{M}^{\mathbb{P}}_{\rm elas} \subset \mathcal{M}_{\rm KdV}$, we will introduce the relative topology in $\mathcal{M}^{\mathbb{P}}_{\rm elas}$ induced from the topology of $\mathcal{M}_{\rm KdV}$.

In §5, we will show another algorithm of explicit computation of solutions of the KdV flow. There we will reconsider the KdV equation in the framework of inverse scattering method and comment the meanings of Theorem 4-2 again at Proposition 5-20. In other words, we will rewrite our second result more analytically. So readers can skip this section except Example 5-21. There we will also review Krichever's construction of algebro-geometrical solutions [Kr, BBEIM] and Baker's original method given about one hundred years ago [Ba2, Ma7]. Using it we showed that there is an injection from the moduli space $\mathfrak{M}_{\rm hyp}$ of hyperelliptic curves to the moduli space $\mathfrak{M}_{\rm KdV}$ of the KdV equation up to an ambiguity; this correspondence enables us to determine function forms of hyperelliptic $\wp$ functions as solutions of the KdV equations for any algebraically given hyperelliptic curves including degenerate curves.

§6 is digression and we will review a result of a loop space over S^2 in the category of topological space **Top**, whose morphism is a continuous map, following the arguments in the textbook of Bott and Tu [BT]. Studies on a loop space in **Top** are well-established and its cohomological properties are well-known. On the other hand, the moduli space of a quantized elastica in $\mathbb{P}$ can be regarded as a loop space in the category of the differential geometry **DGeom**. Thus by loosening the properties in **DGeom** and regarding them as those in **Top**, it is expected that the moduli of a quantized elastica $\mathcal{M}^{\mathbb{P}}_{\rm elas}$ in $\mathbb{P}$ are topologically related to those of a loop space in **Top**. Thus in §6, we will review a loop space in **Top** and show its cohomological properties.

In §7, we will mention the topological properties of the moduli of a quantized elastica $\mathcal{M}^{\mathbb{P}}_{\rm elas}$ and give our third main theorem. As loop spaces in both **Top** and **DGeom** are not finite dimensional spaces when we regard them as manifolds in an appropriate sense, it is not known that de Rham's theorem can be applicable to them. However it is expected that cohomological sequences in both categories should correspond to each other. In other words, it is important to argue existence of functor between triangle categories related to them, *i.e.*, quasi-isomorphism. Precisely speaking, though the closed condition and the reality condition in the moduli $\mathcal{M}^{\mathbb{P}}_{\rm elas}$ make its topological properties difficult to treat, we will tune the low dimensional parts of chain complex of $\mathcal{M}^{\mathbb{P}}_{\rm elas}$ and consider a complex of a quotient spaces $\mathcal{C}\mathcal{M}^{\mathbb{P}}_{\rm elas}$. Then we will show existence of a functor between the triangle categories in loop spaces in both **Top** and **DGeom** as our third main theorem at Theorem 7-4. The existence of the functor means the our theory in **DGeom** is justified in topological investigation. We believe that

this result is meaningful to the investigations of the loop space.

§8 gives the remarks and comments upon our results. First we will comment upon sequences of homotopy of loop spaces in both **Top** and **DGeom**. Next, we will give a possibility of computations of the partition function of a quantized elastica in $\mathbb{C}$. Even in the quantized system, we will show that the orbit space is meaningful, whereas it is well-known that in noncommutative space, concepts of orbit and geometry are sometimes nonsense [C]. So we will comment upon the fact. Further we will remark the relations between our system and Painlevè equation of the first kind [Mat2, In], and between our system and conformal field theory. Finally we will comment upon our results from the a point of view recent progress of Dirac operator related to immersion object based upon [Mat1-6]. We will also mention possibility of higher dimensional case of our consideration there.

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Notations

$\mathbb{R}$ and $\mathbb{C}$ are real and complex number fields respectively. $\mathbb{R}_{\geq 0}$ is the set of the non-negative real numbers. $\mathbb{Z}$ are the set of integers and $\mathbb{N}$ is the set of natural numbers $1, 2, 3, \cdots$. $\mathbb{Z}_{\geq 0}$ is the set of the non-negative integers. $\mathcal{C}^\infty(A, B)$ means the set of B-valued smooth functions over A. $R[x_1, \cdots, x_n]$ is the set of polynomial of $x_1, \cdots, x_n$ with R valued coefficients and $R[[x_1, \cdots, x_n]]$ is the set of formal power series of $x_1, \cdots, x_n$ with R valued coefficients. Others important quantities are listed as follows.

$\mathbb{M}^{\mathbb{P}}$:	Moduli of Loops in $\mathbb{P}$	Defintion 2-4
$\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}$:	Moduli of Loops in $\mathbb{C}^2\setminus\{0\}$, $\varpi : \mathbb{M}^{\mathbb{C}^2\setminus\{0\}} \to \mathbb{M}^{\mathbb{P}}$	Defintion 2-4, Remark 2-5
$\{\gamma, s\}_{\mathrm{SD}}$:	Schwarz derivative	Defintion 2-6
$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$:	Moduli of quantized elastica in $\mathbb{P}$	Defintion 2-10
$\mathbb{M}^{\mathbb{C}}_{\mathrm{elas}}$:	Moduli of quantized elastica in $\mathbb{C}$	Defintion 2-10
$\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$:	$\varpi : \mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}} \to \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$	Defintion 2-10, Remark 2-11
$\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$:	$\pi^{\mathbb{P}}_{\mathrm{elas}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \to \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$	Defintion 2-12
$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$:	$\pi^{\mathbb{C}}_{\mathrm{elas}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \to \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$	Defintion 2-12
$\mathcal{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$:	$\pi^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \to \mathcal{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$	Defintion 2-12
$\mathcal{E}[\gamma]$:	energy of elastica in $\mathbb{P}$	Defintion 2-18
$\mathfrak{D}^s$:	Differential ring over $\mathcal{C}^\infty(S^1, \mathbb{C})$	Defintion 3-1
$\mathfrak{E}^s$:	Micro differential ring to $\mathfrak{D}^s$	Defintion 3-1
$\mathcal{V}^\infty$:	$S^1 \times (\prod_{n=1}^\infty \mathbb{R})$	Defintion 3-2
$\overline{\phi}_{\partial_s u, t}$, $\overline{\varphi}_{\partial_s u, t}$:	the KdVH flow	Defintion 3-2, Proposition 3-11
Ω, $\underline{\Omega}$, :	Recursion differenital operator	Defintion 3-2, Lemma 3-6
$\underset{\mathrm{KdVHf}}{\sim}$:	Equivlent relation related to the KdVH flow	Defintion 3-2
$\phi_{A,t}$:	a flow for A	Defintion 3-7
$\mathfrak{M}^{\mathbb{P}}_{\mathrm{elas}}$:	$\pi_{\mathrm{elas}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \to \mathfrak{M}^{\mathbb{P}}_{\mathrm{elas}} := \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} / \underset{\mathrm{KdVHf}}{\sim}$	Definition 3-18

$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas,finite}}$, $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}g}$:	finite type of the KdV flow and finite g-type flow	Theorem 4-2
$\mathfrak{D}^f$:	Differential ring over $\mathbb{C}[[t_1]]$	Definition 4-4
$\mathfrak{E}^f$:	Micro differential ring to $\mathfrak{D}^f$	Definition 4-4
$\mathfrak{E}^f$:	Micro differential ring with coefficient $\mathbb{C}$	Definition 4-4
$\mathfrak{W}^f$, $\mathfrak{W}^c$:	Subsets of $\mathfrak{E}^f$ and $\mathfrak{E}^c$	Definition 4-4
$\mathcal{L}$:	Subset of $\mathfrak{E}^f$	Lemma 4-8
$\mathcal{A}^f$, $\mathcal{A}^c$:	commutative subrings of $\mathfrak{E}^f$ and $\mathfrak{E}^c$	Lemma 4-8
$\mathfrak{A}^c$:	set of the commutative subrings in $\mathfrak{E}^c$	Definition 4-12
$\mathfrak{D}^t, \mathfrak{E}^t, \mathfrak{W}^t$:	Differential rings over $\mathbb{C}[[t_1, t_2, \cdots]]$	Definition 4-19
$\mathbb{M}_{\mathrm{KdV}}$, $\mathbb{M}_{\mathrm{KdV}}{}^\infty$:	Moduli of the KdV hierarchy	Definition 4-20, Proposition 4-32
$\mathfrak{M}_{\mathrm{KdV}}, g$:	$\pi^g_{\mathrm{KdV}} : \mathbb{M}_{\mathrm{KdV}} \to \mathfrak{M}_{\mathrm{KdV}}, g$	Above Proposition 4-27
$F_g\mathbb{M}_{\mathrm{KdV}}$, $\mathbb{M}_{\mathrm{KdV}g}$:	Filter of Moduli of the KdV hierarchy	Definition 4-26
$\mathfrak{E}^f$:	Micro differential ring with coefficient $\mathbb{C}$	Definition 4-4
$\mathfrak{W}^f{}_g$, $\mathfrak{W}^f{}_{0,1}$:	Gauge freedom	Lemma 4-28
$F_g\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$:	Filter of Moduli of quantized elastica	Proposition 4-29
$\mathfrak{M}_{\mathrm{hyp},g}$:	Moduli of hyperelliptic curves of genus g	Proposition 5-4
$P(X)$:	Path space over X in **Top**	Proposition 6-2
ΩX :	Loop space over X in **Top**	Proposition 6-2
$\mathcal{DM}^{\mathbb{P}}_{\mathrm{elas}}$, $\mathcal{CM}^{\mathbb{P}}_{\mathrm{elas}}$	Complex related to quantized elastica	Proposition 7-1

§2. A Loop in $\mathbb{P}$

In this section we will give an expression of a real curve immersed in a Riemann sphere $\mathbb{P}$ following one of Pedit [Ped]. His expression is based upon the oldest theory of a complex curve embedded in a complex plane $\mathbb{C}$ or an upper half plane $\mathbb{H}$, which was found in ending of the nineteenth century

and studied by Klein, Schwarz, Fuchs, Poincaré and so on [Po]. Using the expression, we will define the moduli space $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ ($\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$) and $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ ($\mathcal{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$) of smooth curves in $\mathbb{P}$ ($\mathbb{C}^2\setminus\{0\}$) in Definition 2-10 and 2-12 and an energy functional of a curve in Definition 2-18, whose integrand is the Schwarz derivative along the curve. As mentioned in Introduction, we will call $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ a moduli space of a quantized elastica.

Let us consider a smooth immersion of a circle into two dimensional complex plane without origin,

$$\psi : S^1 (:= \mathbb{R}/\mathbb{Z}) \hookrightarrow \mathbb{C}^2 \setminus \{0\}, \quad \left(s \mapsto \psi(s) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} \right).$$

Using this map and the natural projection of $\mathbb{C}^2\setminus\{0\}$ to the complex projective space (Riemann sphere) $\mathbb{P}$, we can define the immersion of a loop in $\mathbb{P}$:

2-1. Definition. We define an immersion $\gamma : S^1 \hookrightarrow \mathbb{P}$ by the commutative diagram as $\gamma = \varpi \circ \psi$,

$$\begin{tikzcd} & \mathbb{C}^2 \setminus \{0\} \arrow[d, "\varpi"] \\ S^1 \arrow[r, "\gamma"] & \mathbb{P} \end{tikzcd} .$$

For a chart around $\psi_2 \neq 0$, $s \mapsto \gamma(s) = \dfrac{\psi_1}{\psi_2}(s)$.

2-2. Definition.

(1) The *special linear map* $\mathrm{SL}_2(\mathbb{C}) : \mathbb{C}^2\setminus\{0\} \longrightarrow \mathbb{C}^2\setminus\{0\}$,

$$m \in \mathrm{SL}_2(\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{C} \quad ad - bc = 1 \right\},$$

acts on $\mathbb{P}$ through the Möbius transformation as a symmetric group of $\mathbb{P}$: $g_m : \varpi\circ\psi \mapsto \varpi \circ m\psi$ for $m \in \mathrm{SL}_2(\mathbb{C})$ and for a point $\gamma \in \mathbb{P}$,

$$g_m : \gamma \mapsto \frac{a\gamma + b}{c\gamma + d}, \quad \text{for } m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $\mathrm{PSL}_2(\mathbb{C})$ denote this group including the group action.

(2) Let $\Gamma_0(\mathbb{C})$ denote the subgroup which is characterized by vanishing condition of $(2,1)$-component,

$$\Gamma_0(\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \right\},$$

and $\mathrm{E}_0(\mathbb{C})$ denote its action to $\mathbb{C}\cup\{\infty\}$ using the Möbius transformation.

(3) Let $\Gamma_1(\mathbb{C})$ denote the other subgroup which is characterized by

$$\Gamma_1(\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \;\middle|\; |a| = 1 \right\},$$

and $\mathrm{E}_1(\mathbb{C})$ denote its action to $\mathbb{C}\cup\{\infty\}$ using the Möbius transformation.

2-3. Remark. $\mathrm{PSL}_2(\mathbb{C})$ has following properties:

(1) Translation, rotation and global dilatation: $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_0(\mathbb{C})$, $(b = 1/a)$

$$z \longrightarrow \frac{az+b}{d} = a^2 z + ab.$$

If we restrict the action into $\Gamma_1(\mathbb{C})$, it generates a Euclidean motion induced from $\mathbb{C} = \mathbb{P}\setminus\{\infty\}$.

(2) Coordinate transformation from chart around 0 to chart around ∞:

$$z \longrightarrow -1/z.$$

In Definition 2-10, we give the definitions of moduli spaces of a quantized elastica in $\mathbb{P}$, which are our main objects in this article. However as Proposition 2-8 is correct for a more complicate system, we will give provisional moduli spaces of loops.

2-4. Definition. We define the moduli spaces of loops as sets as follows:

(1)

$$\mathbb{M}^{\mathbb{P}} := \{\gamma : S^1 \hookrightarrow \mathbb{P} \mid \gamma \text{ is smooth immersion}\}/\mathrm{PSL}_2(\mathbb{C}).$$

(2)

$$\mathbb{M}^{\mathbb{C}^2\setminus\{0\}} := \{\psi : S^1 \hookrightarrow \mathbb{C}^2 \setminus \{0\} \mid \psi \text{ is smooth immersion}, \det(\psi(s), \partial_s\psi(s)) = 1 \quad \}/\mathrm{SL}_2(\mathbb{C}).$$

Here $\partial_s := d/ds$.

2-5. Remark.

(1) Let $[\gamma]$ denote an element in $\mathbb{M}^{\mathbb{P}}$ for a representative element $\gamma \in \mathbb{P}$ and an element in $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}$ by $[\psi]$ for a representative element $\psi \in \mathbb{C}^2 \setminus \{0\}$.

(2) For a free loop space $\mathbb{M}$ over a base space M,

$$\mathbb{M} := \{\delta : S^1 \to M \mid \delta \text{ is smooth immersion}\},$$

we can define an evaluation map ev from $S^1 \times \mathbb{M}$ to M by $\mathrm{ev}(s, \delta) = \delta(s)$ [Br]. For $\mathbb{M}^\circ$ ($\circ$ is $\mathbb{P}$ or $\mathbb{C}^2 \setminus \{0\}$), we have the evaluation map whose image is a little bit complicate.

(3) For loops ψ_1 and ψ_2 in $\mathbb{C}^2 \setminus \{0\}$ such that $[\psi_1] = [\psi_2] \in \mathbb{M}^{\mathbb{C}^2\setminus\{0\}}$, we obviously obtain $[\varpi\psi_1] = [\varpi\psi_2]$ in $\mathbb{M}^{\mathbb{P}}$. Thus we also use the notation of ϖ as the map,

$$\varpi : \mathbb{M}^{\mathbb{C}^2\setminus\{0\}} \to \mathbb{M}^{\mathbb{P}}.$$

2-6 Definition. (Schwarz derivative) [Po] $\{\gamma(s), s\}_{\mathrm{SD}}$ is called *Schwarz derivative*, which is defined for a smooth map $\gamma : S^1 \to \mathbb{P}$ equipped with a parameter $s \in S^1$ by,

$$\{\gamma(s), s\}_{\mathrm{SD}} := \partial_s \left(\frac{\partial_s^2 \gamma(s)}{\partial_s \gamma(s)}\right) - \frac{1}{2}\left(\frac{\partial_s^2 \gamma(s)}{\partial_s \gamma(s)}\right)^2.$$

We write it by $\{\gamma, s\}_{\mathrm{SD}}$ or $\{\gamma, s\}_{\mathrm{SD}}(s)$ for brevity.

By elementally computations, the Schwarz derivative is also expressed by

$$\{\gamma, s\}_{\mathrm{SD}} = \left(\frac{\partial_s^3 \gamma(s)}{\partial_s \gamma(s)}\right) - \frac{3}{2}\left(\frac{\partial_s^2 \gamma(s)}{\partial_s \gamma(s)}\right)^2.$$

Straightforward computations give following lemma.

2-7. Lemma. [Po]

(1) *For the action of $g \in \mathrm{PSL}_2(\mathbb{C})$, the Schwarz derivative $\{\gamma, s\}_{\mathrm{SD}}$ is invariant:*

$$\{\gamma, s\}_{\mathrm{SD}} = \{g(\gamma), s\}_{\mathrm{SD}}.$$

(2) *For a diffeomorphism $s' \in \mathrm{Diff}^+(S^1)$*

$$\{\gamma, s\}_{\mathrm{SD}} = (\partial_s s')^2 \left(\{\gamma, s'\}_{\mathrm{SD}} - \{s, s'\}_{\mathrm{SD}}\right)$$

and for $\mathrm{U}(1)$ action on S^1, i.e., $s' = s + \alpha$,

$$\{\gamma(s), s\}_{\mathrm{SD}} = \{\gamma(s' - \alpha), s'\}_{\mathrm{SD}}$$

2-8. Definition/Proposition. [Po] *There is a natural one-to-one correspondence between $\mathbb{M}^{\mathbb{C}^2 \setminus \{0\}}$ and $\mathbb{M}^{\mathbb{P}}$ with the following properties.*

(1) *If $[\gamma]$ is an element of $\mathbb{M}^{\mathbb{P}}$, there exists a unique lifted curve $[\psi]$ as an inverse of the map ϖ ($\varpi[\psi] = [\gamma]$). Let the correspondence be denoted by $\widetilde{\sigma}$, i.e., $\widetilde{\sigma} : \mathbb{M}^{\mathbb{P}} \longrightarrow \mathbb{M}^{\mathbb{C}^2 \setminus \{0\}}$, ($[\psi] = \widetilde{\sigma}([\gamma])$). Then we have $\varpi \circ \widetilde{\sigma}([\gamma]) = [\gamma]$ and $\widetilde{\sigma} \circ \varpi([\psi]) = [\psi]$.*

(2) *For a map $\gamma : S^1 \to \mathbb{P}$ representing a point of $\gamma(s) \in \mathbb{P}$, there is a curve ψ in $\mathbb{C}^2 \setminus \{0\}$ as a solution of the differential equation,*

$$(-\partial_s^2 - \frac{1}{2}\{\gamma, s\}_{\mathrm{SD}}(s))\psi(s) = 0,$$

so that ψ defines an element $[\psi] \in \mathbb{M}^{\mathbb{C}^2 \setminus \{0\}}$ and $[\varpi \psi] = [\gamma]$. This algorithm is a realization of $\widetilde{\sigma}$.

Proof. In this proof, we will deal only with representative elements γ and ψ of $\mathbb{M}^{\mathbb{P}}$ and $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}$. First we will check the well-definedness of $\widetilde{\sigma}$ in (2). Without loss of its generality, we use the chart of $\psi_2 \neq 0$ a loop $\psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2 \setminus \{0\}$. Noting the Remark 2-5 (3), the well-definedness means that the lift of the loop $\gamma(S^1) := \varpi\psi(S^1)$ is uniquely ψ up to $\mathrm{SL}_2(\mathbb{C})$. By differentiating $\det(\psi, \partial_s\psi) = 1$ in s, $(\partial_s^2\psi_2)/\psi_2 = (\partial_s^2\psi_1)/\psi_1$. After straightforward computation, for $\gamma = \psi_1/\psi_2$, we obtain the relation, $(\partial_s^2\psi_2)/\psi_2 = -\{\gamma, s\}_{\mathrm{SD}}/2$. Up to $\mathrm{SL}_2(\mathbb{C})$, ψ is identified with a solution of (2). Hence well-definedness is asserted. Further existence of a solution of this equation in (2) is guaranteed by a special solution, $\psi = \begin{pmatrix} \sqrt{-1}\gamma/\sqrt{\partial_s\gamma} \\ \sqrt{-1}/\sqrt{\partial_s\gamma} \end{pmatrix}$, whose $\det(\psi, \partial_s\psi)$ is unit. The property of Wronskian $\det(\psi, \partial_s\psi) = 1$ and the uniqueness of the solutions of a second order differential equation confirms uniqueness of the solution of (2) up to $\mathrm{SL}_2(\mathbb{C})$. Further due to the construction of the solutions, we will consider the effect of $\mathrm{Diff}^+(S^1)$; for $s'(s)$, the Schwarz derivative changes as in Lemma 2-7, and ∂_s' given by the chain rule, $\psi(s') := \psi(s)/\sqrt{\partial_{s'} s}$. Then $\psi_1(s') : \psi_2(s') = \gamma(s')$ and $\gamma(S^1)$ and $\psi(S^1)$ do not depend on the parameterization. Thus (1) and (2) are completely proved. ∎

2-9. Remark. (Poincaré and Schwarz)[Po, Ba1] By the analytical continuation of $s \in S^1$, γ can be complexfied to γ_c. If γ_c is also embedded in $\mathbb{C}$, γ_c^{-1} is automorphic function. (In general, even though γ is immersed or embedded in $\mathbb{P}$, γ_c can not be immersed in $\mathbb{P}$.) For example the case $s = \wp(\xi)$ ($\xi = \wp^{-1}(s) \in X_1$) for $s \in \mathbb{P}$, $\xi \in \mathbb{C}$, $\{\xi, s\}_{\mathrm{SD}}$ is a meromorphic function of s, where $\wp(\xi)$ is the Weierstrass elliptic function and $X_1 = \mathbb{C}/\Lambda$ is an elliptic curve. These studies are by Klein, Riemann, Poincaré, Schwarz and so on. In this article, we will not restrict ourselves to deal with only with meromorphic function. We will consider transcendental functions of s because our problem is related to a physical problem or an elastica problem as the catenary, another physical curve, is also given by the transcendental function.

In this article, we are concerned with a loop with an energy functional in Definition 2-18. However the integrand $\{\gamma, s\}_{\mathrm{SD}}$ in the energy integration depends upon the parameterization of S^1 or $\mathrm{Diff}^+(S^1)$ from Lemma 2-7. Hence we must fix the parameterizations of the loop in order to treat a loop with the energy functional. Even in $\mathbb{P}$, we can locally define the metric because its tangent space $T\mathbb{P}$ is isomorphic to $\mathbb{C}$ but the action of $\mathrm{PSL}_2(\mathbb{C})$ prevents that the metric becomes global. Hence we restrict ourselves to consider an action of the subgroup $\Gamma_0(\mathbb{C})$ instead of $\mathrm{PSL}_2(\mathbb{C})$.

Let us introduce our main objects in this article.

2-10. Definition. We define the moduli spaces of loops, which are called *moduli of a quantized elastica* or *moduli spaces of a quantized elastica*, as follows:

(1)
$$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} := \{\gamma : S^1 \hookrightarrow \mathbb{P} \mid \text{ smooth immersion, } |\partial_s\gamma(s)| = 1 \}/\mathrm{E}_0(\mathbb{C}).$$

(2)
$$\mathbb{M}^{\mathbb{C}}_{\mathrm{elas}} := \{\gamma : S^1 \hookrightarrow \mathbb{C} \mid \text{smooth immersion, } |\partial_s\gamma(s)| = 1 \}/\mathrm{E}_0(\mathbb{C}).$$

(3)
$$\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}} := \{\psi : S^1 \hookrightarrow \mathbb{C}^2 \setminus \{0\} \mid \text{s smooth immersion, } \det(\psi(s), \partial_s\psi(s)) = 1,$$
$$|\psi_a(s)| = 1 \ (a = 1 \text{ or } 2) \ \}/\Gamma_0(\mathbb{C}).$$

2-11. Remark.

(1) The condition $|\partial_s \gamma(s)| = 1$ means that we will treat only loops with the arc-length parameter s in $\mathbb{C}$ or $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ equipped with the standard flat metric hereafter. We call the condition $|\partial_s \gamma(s)| = 1$ *reality condition* or *arc-length condition*.

(2) We continue to express the elements in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$, $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ by $[\gamma]$ and $[\psi]$ for loops $\gamma \in \mathbb{P}$ and $\psi \in \mathbb{C}^2 \setminus \{0\}$ satisfying appropriate conditions respectively for a while.

(3) Further similar to Remark 2-5 (3), we can define the map from $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ to $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ by ϖ noting that the reality condition $|\partial_s \gamma(s)| = 1$ means $|\psi_a(s)| = 1$ ($a = 1$ or 2) under the condition $\det(\psi(s), \partial_s \psi(s)) = 1$ since $\partial_s \gamma(s) = \det(\psi(s), \partial_s \psi(s))/\psi_2(s)^2$ for $\psi_2(s) \neq 0$.

(4) We can find a representative element by tuning the dilation of $\mathrm{E}_0(\mathbb{C})$. By letting $\oint |d\gamma| = 2\pi$ for a curve with finite length in $\mathbb{C} \cup \{\infty\}$ we have a natural isomorphism,

$$\mathbb{M}^{\mathbb{C}}_{\mathrm{elas}} \approx \mathbb{M}^{\mathbb{C},2\pi}_{\mathrm{elas}} := \{\gamma : S^1 \hookrightarrow \mathbb{C} \cup \{\infty\} \mid \text{ smooth immersion,}$$
$$|\partial_s \gamma(s)| = 1, \ \oint |d\gamma| = 2\pi \ \}/\mathrm{E}_1(\mathbb{C}).$$

Here $|d\gamma| = |\partial_s \gamma(s)| ds$.

(5) Using (1), we have a decomposition whether the length is finite or not, *i.e.*,

$$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \approx \mathbb{M}^{\mathbb{C},2\pi}_{\mathrm{elas}} \coprod \mathbb{M}^{\infty}_{\mathrm{elas}}.$$

This picture is also asserted if one considers a smooth loop in a two-sphere S^2 and its stereographic projection.

For $\mathbb{M}^{\infty}_{\mathrm{elas}}$, we have another representation element,

$$\mathbb{M}^{\infty}_{\mathrm{elas}} \approx \mathbb{M}^{\infty,cvtr}_{\mathrm{elas}} := \{\gamma : S^1 \hookrightarrow \mathbb{C} \cup \{\infty\} \mid \text{ smooth immersion,}$$
$$|\partial_s \gamma(s)| = 1, \ \sup_{s \in S^1} |\partial_s \log \partial_s \gamma(s)| = 1 \ \}/\mathrm{E}_1(\mathbb{C}).$$

Using the equivalences and such a representative element, we can introduce scale in our system.

Next let us introduce smaller moduli spaces for later convenience. Even under the reality condition $|\partial\gamma(s)| = 1$, there is a freedom to choose its origin of the loop, which is denoted by $\mathrm{Isom}(S^1) = \mathrm{U}(1)$. Thus let us define smaller sets with projections to these sets of moduli spaces, *e.g.*, $\pi^{\mathbb{P}}_{\mathrm{elas}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \to \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}/\mathrm{Isom}(S^1)$.

2-12. Definition. (Moduli of a quantized elastica) We define moduli spaces of loops, which are also called *moduli of a quantized elastica*, or *moduli spaces of a quantized elastica*, as follows:

(1) $\pi^{\mathbb{P}}_{\mathrm{elas}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \to \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} := \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}/\mathrm{Isom}(S^1)$.

(2) $\pi^{\mathbb{C}}_{\mathrm{elas}} : \mathbb{M}^{\mathbb{C}}_{\mathrm{elas}} \to \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} := \mathbb{M}^{\mathbb{C}}_{\mathrm{elas}}/\mathrm{Isom}(S^1)$.

(3) $\pi^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}} : \mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}} \to \mathcal{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}} := \mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}/\mathrm{Isom}(S^1)$.

In physics, we are concerned only with the shape of elastica. $\mathcal{M}^{\circ}_{\mathrm{elas}}$ is more important than $\mathbb{M}^{\circ}_{\mathrm{elas}}$. Further we remark here that we have a natural isomorphism $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} \approx \mathbb{M}^{\mathbb{P}}/\mathrm{Diff}^+(S^1)$ as a connection between $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and $\mathbb{M}^{\mathbb{P}}$.

Next we give our correspondence between $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ based on Proposition 2-8

2-13. Definition/Proposition.

(1) *There is a natural one-to-one and continuous correspondence between $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ and $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ with the following properties.*

(1-1) *If $[\gamma]$ is an element of $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$, there exists a unique lifted curve $[\psi]$ in $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ as an inverse of the map ϖ, ($\varpi[\psi] = [\gamma]$). Let the correspondence be denoted by*

$$\sigma : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \longrightarrow \mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}, \quad ([\psi] = \sigma([\gamma])).$$

Then we have $\varpi \circ \sigma([\gamma]) = [\gamma]$ and $\sigma \circ \varpi([\psi]) = [\psi]$.

(1-2) *For a curve $\gamma(s) \in \mathbb{P}$ representing $[\gamma] \in \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$, there is a curve ψ in $\mathbb{C}^2 \setminus \{0\}$ as a solution of the differential equation,*

$$(-\partial_s^2 - \frac{1}{2}\{\gamma, s\}_{\mathrm{SD}}(s))\psi(s) = 0,$$

so that ψ uniquely defines an element $[\psi]$ in $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ and $\varpi[\psi] = [\gamma]$. This algorithm is a realization of σ.

(2) *There is a natural one-to-one correspondence between $\mathcal{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ and $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ induced from σ and ϖ.*

(3) $\mathbb{M}^{\mathbb{C}}_{\mathrm{elas}}$ *and* $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ *($\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$ and $\mathcal{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$) are connected with*

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} = \varpi\mathcal{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}, \quad \mathbb{M}^{\mathbb{C}}_{\mathrm{elas}} = \varpi\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}.$$

Proof. (1) and (2) are essentially the same as the proves in Proposition 2-8 if we check the compatibility between $|\partial_s\gamma(s)| = 1$ and $|\psi_2(s)| = 1$, and continuity of the map. Due to Remark 2-11 (3), the condition $|\psi_2(s)| = 1$ in $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ essentially means the reality condition $|\partial_s\gamma(s)| = 1$ of the map γ on the chart around $\psi_2 \neq 0$.

Let us consider the continuity. For a loop in $\mathbb{P}$, $\gamma'(s) := \gamma(s) + \epsilon v(s)$ with small number ϵ and an element v of $\mathcal{C}^\infty(S^1, \mathbb{C})$, the Schwarz derivative changes

$$\{\gamma', s\}_{\mathrm{SD}} = \{\gamma, s\}_{\mathrm{SD}} + \epsilon\left[\frac{\partial_s^3 v}{\partial_s\gamma} - \frac{\partial_s^3\gamma}{(\partial_s\gamma)^2}\partial_s v - 3\left(\frac{\partial_s^2}{\partial_s\gamma}\right)\left(\frac{\partial_s^2 v}{\partial_s\gamma} - \frac{\partial_s^2\gamma}{(\partial_s\gamma)^2}\partial_s v\right)\right].$$

From the proof of Proposition 2-8, a solution of the its differential equation, ψ', is periodic and thus is a loop in $\mathbb{C}^2 \setminus \{0\}$. For sufficiently small ϵ, the second term becomes small enough. Then using the perturbation theory, we have $\psi' = \psi + \epsilon\eta$ as a solutions of

$$\partial_s^2\psi' + \frac{1}{2}\{\gamma', s\}_{\mathrm{SD}}\psi' = 0.$$

We note $\{\gamma', s\}_{\mathrm{SD}}$ is also invariant for $\mathrm{PSL}_2(\mathbb{C})$. For the condition $|\psi_2| = 1$, we replace the parameter s by s' using the fact in the proof of Proposition 2-8. On the other hand, for $\psi' = \psi + \epsilon\eta'$ we can find $v \in \mathcal{C}^\infty(S^1, \mathbb{C})$ such that $\gamma' := \psi_1'/\psi_2' = \psi_1/\psi_2 + \epsilon v'$. Hence both maps are continuous. ∎

Here we will consider the natural neighborhood in the moduli space $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$.

2-14. Corollary.

(1) *There is a continuous injective map from* $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ *to the function space* $\mathcal{C}^\infty(S^1,\mathbb{C})$.

(2) $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ *has a natural topology generated by neighborhood in the function space* $\mathcal{C}^\infty(S^1,\mathbb{C})$.

Proof. (2) is obvious if (1) is proved. For a given $u \in \mathcal{C}^\infty(S^1,\mathbb{R})$ the functions $\psi \in \mathcal{C}^\infty([0,2\pi),\mathbb{C}^2)$ satisfying

$$(-\partial_s^2 - u)\psi = 0$$

is uniquely determined up to $\mathrm{SL}_2(\mathbb{C})$. In general, even though u is periodic and a function over S^1, ψ is not periodic due to Floquet theorem [MM]. However if ψ is periodic for some u, by letting $|\psi_2| = 1$, $\gamma \in \mathbb{P}$ is uniquely determined by $\gamma = \psi_1/\psi_2$ up to $\mathrm{E}_0(\mathbb{C})$. We note that for such γ and u, u is given as $u = \{\gamma, s\}_{\rm SD}/2$ and $\{\gamma, s\}_{\rm SD}/2$ is invariant for the action of $\mathrm{E}_1(\mathbb{C})$. (For an action $\mathrm{Diff}^+(S^1)$ to the reparameterization of the coordinate s, $\{\gamma, s\}_{\rm SD}/2$ is not invariant and $\{\gamma, s\}_{\rm SD}/2$ changes its value. However we have considered only the arc-length parameterization of s as $|\partial_s\gamma(s)| = 1$.) Hence if ψ is periodic for some u, by letting $|\psi_2| = 1$, it determines $[\gamma] \in \mathbb{M}^{\mathbb{P}}_{\rm elas}$. The continuity is obvious from the previous proposition. ∎

As the injective map in the Corollary 2-14 is a continuous map, the above neighborhood can be geometrically interpreted as a neighborhood of γ.

2-15. Remark. It is known that the free loop space can be a metric space and has natural topology if the base space is a Riemannian manifold [Br, McLau]. As we can regard that an element in $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ with finite length $\int |d\gamma(s)| < \infty$ can be represented by a loop with 2π length whose gravity center exists at the origin of $\mathbb{C}$, it is not difficult to treat the quotient by E_0 or E_1. Consider the image C of the map $\gamma : S^1 \to \mathbb{P}$; $C := \gamma(S^1)$. For such a loop $C \subset \mathbb{P}$ whose represents a point $[C] \in \mathcal{M}^{\mathbb{P}}_{\rm elas}$, there is a normal bundle characterized by the exact sequence of tangent bundle of C and $\mathbb{P}$,

$$0 \to TC \to T\mathbb{P}|_C \to N_C \to 0.$$

Any elements in $T\mathbb{P}|_C$ are decomposed to $T\mathbb{P}|_C \equiv N_C \oplus TC$. For a smooth section $v \in \mathcal{C}^\infty(C, T\mathbb{P}|_C)$ of $T\mathbb{P}|_C$ over C, and for an infinitesimal real parameter ϵ, we have $C + \epsilon v$ as a loop in $\mathbb{P}$. Here $+$ means the natural addition in the local chart $\mathbb{C}$ of $\mathbb{P}$ in the sense of euclidean geometry. Of course, it is important to check whether such a loop is in a different point in $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ or not but if it is, we can find an infinitesimal path from $[\gamma]$ to $[\gamma + \epsilon v_\gamma]$ in $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ by letting $v_\gamma \in \mathcal{C}^\infty(S^1, T\mathbb{P}|_{\gamma(S^1)})$, $v_\gamma := v \circ \gamma$. $|\partial_s(\gamma(s) + \epsilon v_\gamma(s)| = 1$ is not difficult to be treated by reparameterizing s by s' in primitive sense. Further even for the case $[\gamma] = [\gamma + \epsilon v_\gamma]$, we can regard it as a trivial path. If $[\gamma]$ and $[\gamma + \epsilon v_\gamma]$ are different points for an infinitesimal small ϵ, we can regard such $[v_\gamma]$ as an element in a set of smooth sections of tangent bundle of $\mathbb{M}^{\mathbb{P}}_{\rm elas}$, $\mathcal{C}^\infty(T_{[\gamma]}\mathbb{M}^{\mathbb{P}}_{\rm elas}) \equiv \mathcal{C}^\infty(\mathbb{M}^{\mathbb{P}}_{\rm elas}, T_{[\gamma]}\mathbb{M}^{\mathbb{P}}_{\rm elas})$.

We show that there exist such different points. From Corollary 2-14, $\mathcal{C}^\infty(T_{[\gamma]}\mathbb{M}^{\mathbb{P}}_{\rm elas})$ is not the empty set. Let us find an element in $\mathcal{C}^\infty(T_{[\gamma]}\mathbb{M}^{\mathbb{P}}_{\rm elas})$ for each $[\gamma] \in \mathbb{M}^{\mathbb{P}}_{\rm elas}$. As the fiber of $T_p\mathbb{P}$ is isomorphic to $\mathbb{C}$, we define a norm in $v \in \mathcal{C}^\infty(S^1, T\mathbb{P}|_{\gamma(S^1)})$ by sup-norm. (In our article, our argument does not strongly depend upon the norm in $\mathcal{C}^\infty(S^1, T\mathbb{P}|_{\gamma(S^1)})$.) As it is difficult to define a length in scaleless space, we might consider an element in $\mathbb{M}^{\mathbb{C},2\pi}_{\rm elas}$ rather than $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ due to Remark 2-11. Consider $[\gamma] \in \mathbb{M}^{\mathbb{C},2\pi}_{\rm elas}$ for $\gamma : S^1 \to \mathbb{P}$ satisfying the reality condition $|\partial_s\gamma(s)| = 1$ and $\int d\gamma = 2\pi$, and $v \in \mathcal{C}^\infty(S^1, T\mathbb{P}|_{\gamma(S^1)})$. Suppose that $\gamma(S^1) + \epsilon v(S^1)$ preserves local and total length of $\gamma(S^1)$ for sufficiently small ϵ, *i.e.*, $|\partial_s(\gamma(S^1) + \epsilon v(S^1))| = 1$ and $\int d(\gamma + \epsilon v) = 2\pi$ are satisfied. We call the deformation as *isometric*. Then we regard $[\gamma + \epsilon v]$ as an element in $\mathbb{M}^{\mathbb{C},2\pi}_{\rm elas}$. If the vector field $v(\notin \{\text{Euclidean move}\})$ is the isometric deformation, $[\gamma + \epsilon v]$ is a different point from $[\gamma]$ in $\mathbb{M}^{\mathbb{C},2\pi}_{\rm elas}$ and thus they are different points in $\mathbb{M}^{\mathbb{P}}_{\rm elas}$. Then we can naturally define a neighborhood around a loop with finite length in our moduli space $\mathbb{M}^{\mathbb{P}}_{\rm elas}$.

Similarly for an element in $\mathbb{M}^{\infty}_{\mathrm{elas}}$, we can define the neighborhood. For an element $[\gamma]$ in $\mathbb{M}^{\infty,cvtr}_{\mathrm{elas}}$, we define its tangent space and velocity $\mathcal{C}^\infty(S^1, T\mathbb{P}|_{\gamma(S^1)})$. We can constraint the velocity as an isometric path which locally preserves the length. However since $\mathbb{M}^{\infty,cvtr}_{\mathrm{elas}}$ is defined by sup-norm in Remark 2-11, for an element $[\gamma] \in \mathbb{M}^{\infty,cvtr}_{\mathrm{elas}}$ and an isometric path for $v \in \mathcal{C}^\infty(S^1, T\mathbb{P}|_{\gamma(S^1)})$, $[\gamma + \epsilon v]$ generally does not belong to $\mathbb{M}^{\infty,cvtr}_{\mathrm{elas}}$ even with a sufficiently small ϵ. However $[\gamma + \epsilon v]$ is in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and $[\gamma + \epsilon v]$ generates a point in $\mathbb{M}^{\infty}_{\mathrm{elas}}$ again. Hence the path is well-defined by local argument.

Accordingly we can naturally define a neighborhood in our moduli space $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$.

Further as we also define a neighborhood in $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ using Proposition 2-13, we define a topology in our moduli spaces $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$. As the topology comes from that in the loop space we call it *topology of loop space* [Br, McLau].

As the topology of loop space is generated by $\mathcal{C}^\infty(T_\gamma \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}})$, we will consider an infinitesimal deformation parameterized by $t \in [0, \epsilon]$ for a sufficiently small ϵ in detail.

2-16. Remark. Due to the arguments in Remark 2-15, we wish to find one parameter family $[\gamma_t]$ in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ such that $\partial_t[\gamma_t]$ belongs to $\mathcal{C}^\infty(T_\gamma \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}})$.

(1) First, we will consider an *isometric deformation* which locally preserves the arc-length of one parameter family of loops immersed in $\mathbb{P}$: $\gamma_\circ : S^1 \times [0, \epsilon] \to \mathbb{P}$, ($\gamma_t(s) := \gamma(s,t) \in \mathbb{P}$) satisfying

$$[\partial_t, \partial_s]\gamma_t(s) = 0.$$

Here $\partial_s := \partial/\partial s$ and $\partial_t := \partial/\partial t$. We call this condition *isometric condition*. Then if $|\partial_s \gamma_{t=0}(s)| = 1$ for $s \in S^1$, $|\partial_s \gamma_t(s)| = 1$ for $(s,t) \in S^1 \times [0, \epsilon]$.

For $(s, [\gamma_t]) \in S^1 \times \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$, ∂_s acts only on S^1 whereas ∂_t acts only on $[\gamma_t] \in \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$. Of course the relation $[\partial_t, \partial_s](s, [\gamma_t]) = 0$ trivially holds. On the other hand, for the evaluation map $\mathrm{ev}(s, [\gamma_t])$ as Remark 2-5 (2), the action of $[\partial_t, \partial_s]$ is not trivial. However by dealing only with the isometric deformation, we can avoid the noncommutativity between a deformation and the evaluation map.

(2) Let us consider one parameter family of loops immersed in $\mathbb{P}$, $\gamma_\circ : S^1 \times [0, \epsilon] \to \mathbb{P}$, given by a differential equation, which the right hand side depends upon γ_t itself. First assume that the differential equation is $\partial_t \gamma_t = f(\gamma_t)$ for a given functional f. In this case, the deformation depends upon the affine coordinate γ_t in $\mathbb{P}$ and it is not invariant for the action of E_1. Further we note that $\partial_s \gamma_t(s)$ is the tangential vector of the circle $\gamma_t(S^1)$ and $\phi := \log \partial_s \gamma_t(s)$ denotes its tangential angle if $|\partial_s \gamma_t(s)| = 1$. Provided that the deformation $\partial_t \gamma_t$ is governed by a function of $\partial_s \gamma_t(s)$ itself, the deformation must depend upon the angle of $\mathbb{C}$ in $\mathbb{P}$ and a euclidean move. Hence they can not be deformations in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and a deformation in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ does not include $\gamma(s)$ and ϕ.

(3) From Lemma 2-8 (1), $u_t(s) \equiv u(s,t) := \{\gamma_t(s), s\}_{\mathrm{SD}}/2$ is a function of $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$. Further $u_t(s)$ depends only on $\partial_s \log(\partial_s \gamma_t(s))$ and $\partial_s^2 \log(\partial_s \gamma(s))$ due to Definition 2-6. We might consider the deformation in an element γ_t in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ through an equation,

$$\partial_t u_t = f(u_t, \partial_s u_t, \partial_s^2 u_t, \cdots, A),$$

for appropriate functional f and function $A \in \mathcal{C}^\infty(S^1 \times [0, \epsilon], \mathbb{C})$; the function A must be invariant for the action of $\mathrm{E}_0(\mathbb{C})$. If u_t is determined at a time t, γ_t can be reconstructed by Proposition 2-13. If there does not appear $\gamma_t(s)$ or $\partial_s \gamma_t(s)$ themselves in right hand side, the deformation is invariant for the action of $\mathrm{E}^1(\mathbb{C})$ for an appropriate A.

Due to the above consideration, we can consider an infinitesimal deformation in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ by $\partial_t u = f(u, \partial_s u, \partial_s^2 u, \cdots, A)$ and $[\partial_t, \partial_s]\gamma(s,t) = 0$.

As we prepared the tools, it is not difficult to deal with the quotient space $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$. From here, let γ itself denote an element of $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ instead of $[\gamma]$ for a loop $\gamma(S^1) \in \mathbb{P}$ satisfying the conditions. Similarly we write $\psi \in \mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ for the sake of simplicity. Further we will consider a *flow* in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$.

2-17. Remark. Let us consider the situation that for a point $\gamma \in \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and its neighborhood U_γ in terms of the loop topology, we can find another point $\gamma' \in U_\gamma$ such that $\gamma' = \gamma + \epsilon v$ for a sufficiently small ϵ and some velocity $v \in \mathcal{C}^\infty(T_\gamma \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}})$. Suppose that by sequentially finding such points, we construct a curve γ_t, $t \in [0,1]$ in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ connecting between a starting point $\gamma_0 = \gamma$ and a terminal point $\gamma_1 = \gamma''$ for some $\gamma'' \in U_\gamma$. Then we may write the velocity as $\partial_t \gamma_t$ at γ_t. In this way, for each point γ in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$, we can define an immersion of $[0,1]$ in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ for a smooth section $v \in \mathcal{C}^\infty(T_\gamma \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}})$ if it is well-defined. We call such a immersion *flow* in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$.

Further for each a point γ_t in a flow, $t \in [0,1]$ with $v_t \in \mathcal{C}^\infty(T_{\gamma_t} \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}})$, let us assume that we can choose another element $v_t' \in \mathcal{C}^\infty(T_{\gamma_t} \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}})$ and find a point $\gamma_{t,\epsilon}$ in the neighborhood of γ_t such that $\gamma_{t,\epsilon} = \gamma_t + \epsilon u'$ for a sufficiently small parameter ϵ. Then we can consider duplex flow such as $\gamma_{t,t'}$ for $[0,1]^2$ in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$. Similarly we can deal with an immersion of $[0,1]^m$ in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$. For the case, we call the immersion γ_t of $t \in [0,1]^m$ *multiple flow.* Further for a certain case, $[0,1]^m \in \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ can be extended to $\mathbb{R}^m \in \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ where m is a positive integer or the infinite number.

Similarly we can deal with flow in $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$. We define the KdV and KdVH flow as an extension of $[0,1]^m$ immersion to $\mathbb{R}^m$ immersion in §3.

2-18. Definition. (Energy of a quantized elastica) We introduce an energy functional of $\gamma \in \mathcal{M}^{\mathbb{C},2\pi}_{\mathrm{elas}} \approx \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$, called *Euler-Bernoulli energy functional*, by

$$\mathcal{E}[\gamma] := \frac{1}{2\pi} \int_{S^1} \{\gamma(s), s\}_{\mathrm{SD}} ds.$$

2-19. Lemma. *For $\gamma \in \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$, the energy $\mathcal{E}[\gamma]$ is non-negative real.*

Proof. The Schwarz derivative can be expressed by

$$\{\gamma, s\}_{\mathrm{SD}} = \partial_s^2 \log(\partial_s \gamma) - \frac{1}{2}(\partial_s \log(\partial_s \gamma))^2.$$

Due to Definition 2-10, the reality condition $|\partial_s \gamma(s)| = 1$, we let $\partial_s \gamma = \exp(\sqrt{-1}\phi)$, ϕ is a real smooth function over S^1, $\phi(0) = \phi(2\pi)$. Hence

$$\int_{S^1} \{\gamma, s\}_{\mathrm{SD}} ds = \int_{S^1} ds \frac{1}{2}(\partial_s \phi)^2,$$

which is real. ∎

2-20. Remark. [Mat2]

(1) By Lemma 2-7, the integrand in the energy $\mathcal{E}$ is invariant for the action of $\mathrm{PSL}_2(\mathbb{C})$. However the diffeomorphism of S^1, $\mathrm{Diff}^+(S^1)$, changes the energy. Hence we cannot find a well-defined energy over $\mathcal{M}^{\mathbb{P}}$.

Further for $\gamma \in \pi^{\mathbb{P}}_{\mathrm{elas}}\mathbb{M}^{\infty,cvtr}_{\mathrm{elas}}$, we can also consider a correspondence

$$\int_{S^1} \{\gamma, s\}_{\mathrm{SD}} ds,$$

by giving up to considering dilatation symmetry. However it is neither well-defined for the dilatation.

As we wish to neglect the problem for $\pi^{\mathbb{P}}_{\mathrm{elas}}\mathbb{M}^{\infty}_{\mathrm{elas}}$, we restrict ourselves to deal with $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$. In other words, we will consider the energy functional only for $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$.

(2) We regard the energy function as a section of line bundle over $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$,

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathrm{Energy}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}) \\ & & \downarrow \\ & & \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} \end{array}$$

.

(3) As mentioned in Introduction, for $\gamma \in \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$, this energy functional $\mathcal{E} = \int_{S^1} \{\gamma, s\}_{\mathrm{SD}} ds$ is identified with $\int_{S^1} (\partial_s^2 \gamma / \partial \gamma)^2 ds$; thus we call it Euler-Bernoulli energy functional. The stationary points of $\mathcal{E}$ in $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ in the meaning (1) were investigated by Euler [E, L, T1, 2]. Even though we will not touch the problem in this paper, we are implicitly considering the partition function of an elastica as a problem of physics [Mat2],

$$Z = \int_{\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}} D\gamma \exp\left(-\beta \int_{S^1} \{\gamma, s\}_{\mathrm{SD}} \mathrm{d}s\right).$$

In order to know this partition function (which is not mathematically still well-defined), we must investigate the moduli space of curve $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} \subset \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ and we will do in this paper.

§3. KdVH flow

Our studies are based upon the discovery of Goldstein and Pertich [GP1, 2] on the MKdV flow for a loop in $\mathbb{C}$ and that of Langer and Perline [LP] on the nonlinear Schrödinger flow for a loop in $\mathbb{R}^3$. Using their results, one of authors studied the moduli of loops in $\mathbb{C}$ [Mat2] and loops in $\mathbb{R}^3$ [Mat4]. Our purpose is to give mathematical implications of these works [Mat2, 4] using results of Pedit [Ped]. In this section, we will give our main theorem 3-4 and its proof, which are of a relation between the moduli of a quantized elastica in $\mathbb{P}$ and the KdV flow.

In order to express the system of the KdV equation, we will introduce the differential algebra and its division algebra before our main arguments in this section.

3-1. Definition. [SN]

(1) The *differential ring* $\mathfrak{D}^s$ is defined by,

$$\mathfrak{D}^s := \{\sum_{k\ge 0}^{N} a_k(s)\partial_s^k \mid N<\infty,\ a_k(s)\in \mathcal{C}^\infty(S^1,\mathbb{C}), s\in S^1\}.$$

(2) The *degree* of a differential operator, $D\in\mathfrak{D}^s$, is denoted by deg$D$,

$$\deg : \mathfrak{D}^s \longrightarrow \mathbb{Z}_{\ge 0},$$

where $\mathbb{Z}_{\ge 0}$ is the set of non-negative integers.

(3) The *micro-differential ring* $\mathfrak{E}^s$ to $\mathfrak{D}^s$ is defined by

$$\mathfrak{E}^s := \{\sum_{k=-\infty}^{N} a_k(s)\partial_s^k \mid N<\infty,\ a_k(s)\in \mathcal{C}^\infty(S^1,\mathbb{C}),\ s\in S^1\},$$

where $\deg : \mathfrak{E}^s \longrightarrow \mathbb{Z}$ and the product is defined by the extended Leibniz rule,

$$\partial_s^n a = \sum_{r=0}^{\infty}\begin{pmatrix} n\\ r\end{pmatrix}(\partial_s^r a)\partial_s^{n-r}, \qquad \begin{pmatrix} n\\ r\end{pmatrix} := \frac{1}{r!}n(n-1)\cdots(n-r+1).$$

(4) The projections $+$ and $-$ are defined by

$$+ : \mathfrak{E}^s \longrightarrow \mathfrak{D}^s, \quad (L\mapsto L_+), \quad - : \mathfrak{E}^s \longrightarrow \mathfrak{E}^s\setminus\mathfrak{D}^s, \quad (L\mapsto L_-, \quad L = L_+ + L_-).$$

Hereafter we will write a map from S^1 to $\mathbb{P}$, $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ by the same γ. Noting Remark 2-17, let us define the KdV and KdVH flows, which satisfy the isometric condition as in Proposition 3-11.

3-2. Definition. (KdV flow and KdVH flow)

(1) The *KdV flow* is defined as the immersion
$\gamma_\circ : \mathbb{R}\hookrightarrow \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and $\psi_\circ : \mathbb{R}\hookrightarrow \mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$, $(t\mapsto(\gamma_t,\psi_t))$,
which satisfies the following properties:
1-1) $\gamma_t(s) = \varpi\circ\psi_t(s)$, for each $t\in\mathbb{R}$.
1-2) $u(s,t) := \{\gamma_t(s), s\}_{\mathrm{SD}}/2$ obeys the KdV equation,

$$\partial_t u + 6u\partial_s u + \partial_s^3 u = 0.$$

If for a point $\gamma\in\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and its corresponding point $\psi\in\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$, there is one of the KdV flows such that $\gamma_t=\gamma$ and $\psi_t=\psi$ for some $t\in\mathbb{R}$, we say that γ or ψ *belongs* to the KdV flow $\gamma_\circ$ or $\psi_\circ$.

(2) Let us introduce a formal infinite dimensional parameter space,

$$\mathcal{V}^\infty := S^1\times(\prod_{n=1}^{\infty}\mathbb{R}), \quad t=(t_1,t_2,t_3,\cdots)\in\mathcal{V}^\infty,\ t_1\in S^1.$$

Then the *KdVH flow* is defined as the immersion

$$(\gamma_\circ, \psi_\circ) \equiv \overline{\phi_{\partial_s u, t}} : \mathcal{V}^\infty \hookrightarrow \mathbb{M}^{\mathbb{P}}_{\text{elas}} \times \mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\text{elas}}, \quad (t \mapsto (\gamma_t, \psi_t)),$$

which satisfy the following conditions:

1-1) $\gamma_t(s) = \varpi \circ \psi_t(s)$,

1-2) $\overline{\phi_{\partial_s u, t}}$ is given by

$$\gamma(s,t) \longrightarrow \gamma(s, t+\delta t) = \exp(\sum_{n=1} \delta t_n \partial_{t_n})\gamma(s,t),$$

whose each t_n deformation obeys the n-th KdV equation ($n \geq 1$),

$$\partial_{t_n} u = -\Omega^{n-1}\partial_s u,$$

where Ω is a micro-differential operator,

$$\Omega = (\partial_s^2 + 2u + 2\partial_s u \partial_s^{-1}) \in \mathfrak{E}^s.$$

If for a point $\gamma \in \mathbb{M}^{\mathbb{P}}_{\text{elas}}$ and its corresponding point $\psi \in \mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\text{elas}}$, there is one of the KdV flows such that $\gamma_t = \gamma$ and $\psi_t = \psi$ for some $t \in \mathcal{V}^\infty$, we say that γ or ψ *belongs* to the KdVH flow $\gamma_\circ$ or $\psi_\circ$.

(3) We define a relation,

$$\gamma \underset{\text{KdVHf}}{\sim} \gamma',$$

for two points $\gamma, \gamma' \in \mathcal{M}^{\mathbb{P}}_{\text{elas}}$ if these γ and γ' are on an orbit of the projection of the KdVH flow $\pi^{\mathbb{P}}_{\text{elas}} \circ \overline{\phi_{\partial_s u, t}}$, *i.e.*, every points in the fibers ${\pi^{\mathbb{P}}_{\text{elas}}}^{-1}\gamma$ and ${\pi^{\mathbb{P}}_{\text{elas}}}^{-1}\gamma'$ belongs the same KdVH flow.

For convenience, let γ_t (ψ_t) denote the KdV flow or KdVH flow instead of $\gamma_\circ$ ($\psi_\circ$) from this.

3-3.

(1) Though the well-definedness of the above definition is later investigated, these flows satisfy the isometric condition as in Proposition 3-11.

(2) We will note that the space $\mathcal{V}^\infty$ has an algebro-analytic structure induced from the equations,

$$\partial_{t_{n+1}} u = -\Omega \partial_{t_n} u.$$

(3) The $n = 2$ KdVH flow obeying $\partial_{t_2} u = \Omega \partial_s u$ is identified with the KdV flow in (1) of Definition 3-2.

Our first main theorem is as follows:

3-4. Theorem.

(1) *The relation* $\underset{\mathrm{KdVHf}}{\sim}$ *in the Definition 3-2 becomes an equivalent relation; for arbitrary* γ *in* $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ *there is one of the KdVH flows to which* γ *belongs, and for* $\gamma \underset{\mathrm{KdVHf}}{\sim} \gamma'$ *and* $\gamma' \underset{\mathrm{KdVHf}}{\sim} \gamma''$*, we have a relation* $\gamma \underset{\mathrm{KdVHf}}{\sim} \gamma''$*. By this relation, we can define an equivalent class*

$$\mathfrak{C}[\gamma] := \{\gamma' \in \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} \mid \gamma' \underset{\mathrm{KdVHf}}{\sim} \gamma\}, \quad \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} = \coprod_{\gamma} \mathfrak{C}[\gamma].$$

Similarly we can define

$$\mathfrak{C}_{\mathbb{C}}[\gamma] := \{\gamma' \in \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} \mid \gamma' \underset{\mathrm{KdVHf}}{\sim} \gamma\}, \quad \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} = \coprod_{\gamma} \mathfrak{C}_{\mathbb{C}}[\gamma].$$

(2) *The KdVH flow conserves the energy* $\mathcal{E}$*. In other words, for the subspace of* $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$*,*

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E} := \left\{ \gamma \in \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} \middle| \ \mathcal{E}[\gamma] - E = 0 \ \right\},$$

and a curve $\gamma \in \mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$*, the following relation holds*

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},\mathcal{E}[\gamma]} \supset \mathfrak{C}_{\mathbb{C}}[\gamma].$$

(3) *The moduli space of a quantized elastica* $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$ *is decomposed as*

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}} = \coprod_{E} \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E}, \quad \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E} = \coprod_{\gamma, \mathcal{E}[\gamma]=E} \mathfrak{C}_{\mathbb{C}}[\gamma].$$

Noting Remark 2-16, we will investigate the moduli spaces $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ by considering flows over there and prove our theorem. Here we mention the strategies of the proof of the theorem.

3-5. We plane to investigate $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ by dealing with a group which is generated by a Lie algebra associated with $T\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$. By the correspondence between $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\mathrm{elas}}$ in Proposition 2-8, we can identify $\gamma(s)$ with $(\psi_1, \psi_2)(s)$. We firstly deal with wider class of flows $\phi_{A,t}$ in Lemma 3-6, which is characterized by a smooth function A over $S^1 \times [0, 1]$. In Lemma 3-9, we find that an arbitrary flow $\phi_{A,t}$ approximately preserves the energy of elastica in Definition 2-18. Due to the argument in Remark 3-10, we choose a special A as $A = \partial_s\{\gamma, s\}_{\mathrm{SD}}$ and then the flow is identified with the KdV flow in Proposition 3-11. As shown in Proposition 3-15, 16, and 17, we use the regular properties of the KdV hierarchy and prove the theorem.

Noting Remark 2-16, we have the following lemma.

3-6. Lemma. (Goldstein-Pertich, Pedit)[GP1, GP2, Ped] *Let us consider a flow of* $[0, \epsilon]$ *for a real number* $\epsilon > 0$*:*

$$[0, \epsilon] \to \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}, \quad (t \mapsto \gamma_t),$$

i.e., it is realized by an isometric deformation,

$$[\partial_s, \partial_t]\gamma_t(s) = 0.$$

(1) *Every isometric deformation $\gamma_t(s)$ locally obeys the equation of motion,*

$$\partial_t u = -\Omega A(s,t),$$

where $u = \{\gamma, s\}_{\rm SD}/2$ and $A(s,t)$ is an appropriate smooth function over $(s,t) \in S^1 \times [0,\epsilon]$.

(2) *For the function $A(s,t)$, there exists a smooth function $B(s,t)$ such that $A(s,t) = -\partial_s B(s,t)/2$ and this equation of motion is locally rewritten by,*

$$\partial_t u = \frac{1}{2}\underline{\Omega} B(s,t),$$

where $\underline{\Omega} := \Omega\partial_s$,

$$\underline{\Omega} = (\partial_s^3 + 2u\partial_s + 2\partial_s u).$$

Proof. Using the one-to-one correspondence between $\mathbb{M}^{\mathbb{P}}_{\rm elas}$ and $\mathbb{M}^{\mathbb{C}^2\setminus\{0\}}_{\rm elas}$, we lift the flow γ_t to $\psi_t := \sigma\gamma_t$. In this proof, we consider representative elements of the image of its evaluation map, $\gamma_t(s)$ and $\psi_t(s)$. Due to the linear independence given by $\det(\partial_s\psi_t(s), \psi_t(s)) = 1$, we express the deformation in terms of $\psi_t(s)$ and $\partial_s\psi_t(s)$;

$$\partial_t\psi_t(s) = (A(s,t) + B(s,t)\partial_s)\psi_t(s),$$

where $A(s,t)$ and $B(s,t)$ are smooth functions over (s,t). However from $\partial_t \det(\psi_t(s), \partial_s\psi_t(s)) = 0$, we have the constraint,

$$\partial_s B(s,t) = -2A(s,t), \tag{3.1}$$

using $[\partial_s, \partial_t]\psi_t(s) = 0$. Noting $u(s,t) = -(\partial_s^2\psi_2(s))/\psi_2(s)$, we perform a straightforward computations of $\partial_t u(t,s)$, we obtain the equation in (1). On the other hand, if the equation is satisfied, we can reduce the equation to $[\partial_t, \partial_s]\gamma_t(s) = 0$. Similarly we obtain (2). ∎

Let us introduce another formal infinite dimensional parameter spaces, $t = (t_1, t_2, t_3, \cdots) \in [0,\epsilon]^\infty$ and a formal multiple flow $\phi_{A,t}$ with the infinite dimensional parameters, which is locally defined.

3-7 Definition. For $t \in [0,\epsilon]^\infty$ for a sufficiently small parameter ϵ, we will define an infinitesimal multiple flow,

$$\phi_{A,t} : [0,\epsilon]^\infty \longrightarrow \mathbb{M}^{\mathbb{P}}_{\rm elas}, \quad (t \mapsto \gamma_t),$$

induced from the formal variation for a sufficiently small δt, ($\delta t_i < \epsilon < N\delta t_i$, a small natural number N) and image of evaluation map $\gamma(s,t) := \gamma_t(s)$,

$$\gamma(s,t) \mapsto \gamma(s, t+\delta t) = \exp(\sum_{n=1} \delta t_n \partial_{t_n})\gamma(s,t) := (1 + \sum_{n=1} \delta t_n\partial_{t_n})\gamma(s,t) + \mathcal{O}(\delta t^2),$$

with local relations,

$$[\partial_s, \partial_{t_n}]\gamma(s,t) = 0, \quad (n \geq 1),$$
$$\partial_{t_n} u = -\Omega^{n-1}A(s,t), \quad (n \geq 1),$$

where $u(s,t) = \{\gamma_t(s), s\}_{\rm SD}/2$, $A(s,t)$ and $B(s,t)$ are appropriate smooth functions over $S^1 \times [0,\epsilon]^\infty$ such that $2A = -\partial_s B$.

3-8 Remark.

(1) In terms of the definition of the exponential function to the base e,

$$\exp(O) = \lim_{N'\to\infty} \left(1 + \frac{O}{N'}\right)^{N'},$$

the development of δt_n generates $[0,\epsilon]^\infty$. By tuning N' compatible to N in Definition 3-7, we can define the exponent action to $\gamma(s,t)$.

(2) If $\Omega^{n-1}A(s,t)$ vanishes for $n > M$ for a natural number M, the deformation is of finite dimensional. Then the flow $\phi_{A,t}$ is well-defined for a sufficiently small ϵ.

(3) In general, the above flow $\phi_{A,t}$ is a formal one and its well-definedness is not guaranteed. However if it is well-defined, it gives an isometric deformation of a curve $\gamma(s) \in \mathbb{M}^{\mathbb{P}}_{\text{elas}}$. In fact due to the relation $\partial_{t_{n+1}} = \Omega\partial_{t_n}$, we have the flow

$$\partial_{t_n}\psi_t(s) = (A_n + B_n\partial_s)\psi_t(s),$$

where $A_2 = A = -\partial_s B/2$, $B_2 = B$, $A_1 = \Omega^{-1}A$, and

$$A_n = \Omega A_{n-1}, \qquad \partial_s B_n = \Omega B_{n-1}, \quad (n \ge 2).$$

Then the above relation $\partial_{t_n} u = -\Omega^{n-1}A(s,t)$ turns out to be the standard type of the flow for A_n in Lemma 3-6.

3-9. Lemma.

For $\gamma \in \mathbb{M}^{\mathbb{C}}_{\text{elas}}$ and $A \in \mathcal{C}^\infty(S^1\times[0,\epsilon]^\infty, \mathbb{C})$, the infinitesimal flow $\phi_{A,t}$ preserves the energy functional modulo $(\delta t)^2$;

$$\frac{1}{2\pi}\int_{S^1} \{\gamma_t, s\}_{\text{SD}}\ ds = \frac{1}{2\pi}\int_{S^1} \{\gamma_{t+\delta t}, s\}_{\text{SD}}\ ds + \mathcal{O}((\delta t)^2).$$

Proof. Noting Remark 3-8 and by Eq.(3.1) in the proof of Lemma 3-6, we have the relations,

$$\partial_s B_n = -2A_n = -2\Omega A_{n-1}, \quad \partial_s B_n = \Omega B_{n-1}.$$

When we will apply the relation to the right hand side of the lemma, $(u(s,t) := \{\gamma_t, s\}_{\text{SD}}/2)$,

$$\begin{aligned}
\int_{S^1} u(s,t+\delta t)ds &= \int_{S^1} u(s,t)ds + \sum_{n=1} \delta t_n \int_{S^1} \partial_{t_n} u(s,t)ds + \mathcal{O}((\delta t)^2)\\
&= \int_{S^1} u(s,t)ds - \sum_{n=2} \delta t_n \int_{S^1} \Omega A_n ds + \frac{1}{2}\int_{S^1} \partial_s B ds + \mathcal{O}((\delta t)^2)\\
&= \int_{S^1} u(s,t)ds + \frac{1}{2}\sum_{n} \delta t_n \int_{S^1} \partial_s B_{n+1}(s,t)ds + \mathcal{O}((\delta t)^2)\\
&= \int_{S^1} u(s,t)ds + \mathcal{O}((\delta t)^2).
\end{aligned}$$

We completely prove the lemma. ∎

3-10. Remark.

(1) This flow $\phi_{A,t}$ could be regarded as an infinitesimal action of a diffeomorphism of $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$, which is a (infinite dimensional) Lie group G_A if it can be well-defined.

(2) We can regard S^1 as a Riemannian manifold with a metric ds^2. Then ∂_s is the Killing vector and $\exp(\sqrt{-1}s)$ is a geodesic flow. They are a generator and an element of the $\mathrm{Isom}(S^1)$ =U(1) group respectively;

$$\mathrm{U}(1) : S^1 \longrightarrow S^1, \quad (\exp(\sqrt{-1}s) \mapsto \exp(\sqrt{-1}(s + s_0)),$$

For $g_0 \in \mathrm{U}(1)$, g_0 gives a natural automorphism of $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$.

(3) Since there is the natural projection $\pi^{\mathbb{P}}_{\mathrm{elas}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \to \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$, the U(1) action on $\gamma \in \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ must be trivial $g_0\gamma = \gamma$ for $g_0 \in$U(1) and we have the relation $g_0 \circ \pi^{\mathbb{P}}_{\mathrm{elas}} = \pi^{\mathbb{P}}_{\mathrm{elas}} \circ g_0$. It implies that the immersion of the loop S^1 is consistent with U(1) action.

(4) For a curve $\gamma(s) \in \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$, we can locally express the U(1) action,

$$(\partial_s - \partial_{s_0})\{\gamma, s\}_{\mathrm{SD}}(s, s_0) = 0, \quad (\partial_s - \partial_{s_0})\gamma(s, s_0) = 0.$$

These equations faithfully represent the U(1) symmetry or translation, $\gamma(s) \to \gamma(s - s_0)$ in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$.

(5) Due to the above remarks, if exists, G_A should include G_0 =U(1) as its normal subgroup. Accordingly it is natural that A in Definition 3-7 starts with the internal symmetry: $A = \partial_s u$ and $\partial_{t_1} u = \partial_s u$ for $u = \{\gamma, s\}_{\mathrm{SD}}$.

(6) When we consider the multiple flow generated by $\phi_{\partial_s u,t}$ ($A = \partial_s u$), it means that we deal with the variation,

$$\gamma(s, t) \longrightarrow \gamma(s, t + \delta t) = \exp(\sum_n \delta t_n \partial_{t_n})\gamma(s, t),$$

which obeys

$$\partial_{t_n} u = -\Omega^{n-1}\partial_s u.$$

Following Definition 3-2, they are locally identified with the KdVH flow.

(7) Due to the Remark 2-16, this multiple flow is locally well-defined in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$.

(8) Physically speaking for the above arguments, we are implicitly investigating a partition function of a "elastic" curve in $\mathbb{P}$. We require that the partition function must naturally include classical shapes whose have the above trivial translation symmetry as the Goldstone bosons or the Jacobi fields [R]. This requirement makes the group structure acting $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ (if exists) contain this trivial symmetry [Mat2].

We will summarize the above results as a proposition.

3-11. Proposition.

(1) *The multiple flow $\phi_{\partial_s u,t}$ contains a subflow $\phi_{\partial_s u,t_1}$ generated by*

$$(\partial_s - \partial_{s_0})\{\gamma, s\}_{\mathrm{SD}} = 0.$$

This domain of $t_1 \in [0, \epsilon]$ is extended to S^1 and is consistent with the projection $\pi^{\mathbb{P}}_{\mathrm{elas}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \to \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$, i.e., there exists $\varphi_{\partial_s u,t}$ such that $\pi^{\mathbb{P}}_{\mathrm{elas}} \circ \phi_{\partial_s u,t} = \varphi_{\partial_s u,t} \circ \pi^{\mathbb{P}}_{\mathrm{elas}}$.

(2) *By choosing $A = \partial_s u$ for $u = \{\gamma, s\}_{\mathrm{SD}}/2$, the flow $\phi_{\partial_s u,t}$ defined in Definition 3-7 is well-defined as a flow in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ and can extend the domain of the flow $[0, \epsilon]^\infty \to \mathcal{V}^\infty$.*

(3) *$\phi_{\partial_s u,t}$ is identified with the KdVH flow $\overline{\phi_{\partial_s u,t}}$ by extending $[0, \epsilon]^\infty$ to $\mathcal{V}^\infty$.*

(4) *There exists a flow* $\overline{\varphi_{\partial_s u,t}}$ *in* $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ *such that* $\pi^{\mathbb{P}}_{\mathrm{elas}} \circ \overline{\phi_{\partial_s u,t}} = \overline{\varphi_{\partial_s u,t}} \circ \pi^{\mathbb{P}}_{\mathrm{elas}}$*, we also call it KdVH flow.*
(5) *For the KdVH flow, we have algebraic relations among multi-times* t_n *as* $\partial_{t_{n+1}} u = \Omega \partial_{t_n} u$.
(6) *The KdVH flow preserves the decomposition in Remark 2-11.*
(7) *The restricted flow of the KdVH flow to* $\mathbb{M}^{\mathbb{C}}_{\mathrm{elas}}$ *preserves the energy functional exactly.*

Proof. (1) is obvious from Remark 3-10. If (2) is satisfied, (3), (4) and (5) are naturally given from Remark 3-10. Since the KdVH flow consists of isometric deformations, (6) is obvious. (2) and (7) will be asserted by Propositions 3-15 and 3-16. ∎

Firstly we note that (7) should be compared with the Lemma 3-9. Next we also note that in order to prove (2), we should check 1) the well-definedness of the KdVH flow locally and 2) the extension of the domain to $\mathcal{V}^\infty$. If the well-definedness of the KdVH flow is guaranteed, we can find the neighborhood of a point $\gamma \in \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ by the KdVH flow to γ as its initial state, because the KdVH flow consists of the isometric deformations. We can consider the process in $\mathbb{M}^{\mathbb{C},2\pi}_{\mathrm{elas}}$ as mentioned in Remark 2-15.

Here we will introduce the words of a dynamic system here apart from our notations in main subject [AM].

3-12. Definition. [AM, Br] We will consider a manifold M equipped with a closed real 2-form ω. We will use the notations: $i_Y v$ is the interior product of a vector field Y and a differential form v.
(1) A vector field Y is called *symplectic* if $i_Y \omega$ is closed.
(2) A vector field Y is called a *Hamiltonian vector field* if there exists a function f such that $i_Y \omega = df$.

Corresponding to Definition 3-12, we will define quantities in the KdV flow in Definition 3-13 and give Proposition 3-15 by assuming $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ as a (infinite dimensional) manifold [AM].

3-13. Definition. [AM]
(1) In our KdVH flow, we define a 2-form ω for vectors Y_1 and Y_2 over $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$,

$$\omega(Y_1, Y_2) := \frac{1}{2} \int_{S^1} \left(\int_0^s (Y_2(s) Y_1(s') - Y_1(s) Y_2(s')) ds' \right) ds.$$

(2) We define the quantities X_n and h_n and variation $\overline{\delta}/\overline{\delta} u$ for the KdVH flow: $h_0 = u/2$, $X_0 = 0$ and

$$X_n(u) := \Omega^{n-1} \partial_s u, \quad X_n(u) = \partial_s \frac{\overline{\delta} h_n}{\overline{\delta} u},$$

where

$$\frac{\overline{\delta} h_n}{\overline{\delta} u} = \frac{\delta h_n}{\delta u} - \partial_s \frac{\delta h_n}{\delta(\partial_s u)} + \partial_s^2 \frac{\delta h_n}{\delta(\partial_s^2 u)} - \partial_s^3 \frac{\delta h_n}{\delta(\partial_s^3 u)} + \cdots .$$

The existence of such h_n will be guaranteed in Proposition 4-18. Noting $\underline{\Omega} = \Omega \partial_s$, and from the definition we have a recursion relation,

$$\partial_s \frac{\overline{\delta} h_n}{\overline{\delta} u} = \underline{\Omega} \frac{\overline{\delta} h_{n-1}}{\overline{\delta} u},$$

if h_n exists. In Proposition 4-18, we show existence of a set of functionals $\overline{h}_n = \mathrm{res}\dfrac{2^{2n}}{2n-1}L^{(2n-1)/2}$, u satisfying $\partial_s\overline{h}_n = \Omega\partial_s\overline{h}_{n-1}$ with $\overline{h}_1 = u/2$. Here $L = \partial_s^2 + u$ and "res" means the coefficient of the ∂_s^{-1} in the notations in §4. In other words, $\partial_s\overline{h}_n = \Omega^n\partial_s\overline{h}_1 = \Omega^n\partial_s u/2$. Further from the definition, we have [D]

$$\int \frac{\overline{\delta h_n}}{\overline{\delta} u} ds \equiv 2(2n-3)\int \overline{h}_{n-1} ds,$$

since $\displaystyle\int \frac{\overline{\delta h_n}}{\overline{\delta} u} \equiv \int \frac{\delta \overline{h}_n}{\delta u}$ due to periodicity and

$$\begin{aligned}
\int \frac{\overline{\delta}}{\overline{\delta} u}\mathrm{res}(L^{r/2})ds &= \int \mathrm{res}\left(\sum_{i=1}^{r-1}(L^{1/2})^i\frac{\overline{\delta} L^{1/2}}{\overline{\delta} u}(L^{1/2})^{r-i-1}\right)ds \\
&= r\int \mathrm{res}(L^{(r-1)/2}\frac{\overline{\delta} L^{1/2}}{\overline{\delta} u})ds \\
&= \frac{r}{2}\int \mathrm{res}(L^{(r-2)/2}\frac{\overline{\delta} L}{\overline{\delta} u})ds \\
&= \frac{r}{2}\int \mathrm{res}(L^{(r-2)/2})ds.
\end{aligned}$$

Let $h_n \equiv 2^n\overline{h}_{n-1}/(2n+1)$ modulo periodic functions and $X_n = \partial_s\overline{h}_n$ with $\overline{h}_0 = 0$. Hence Definition 3-13 is guaranteed by Proposition 4-18.

Here we give the vector fields X_n and quantities h_n explicitly:

3-14. Example. (KdVH flow)

$$\begin{array}{lcr}
n=0: & X_0(u) = 0, & h_0 = \frac{1}{2}u \\
n=1: & X_1(u) = \partial_s u, & h_1 = \frac{1}{2}u^2 \\
n=2: & X_2(u) = \partial_s(3u^2 + \partial_s^2 u), & h_2 = u^3 + \frac{1}{2}(\partial_s u)^2 \\
n=3: & X_3(u) = \partial_s(10u^3 + 5(\partial_s u)^2 + 10u\partial_s^2 u + \partial_s^4 u), & h_3 = \frac{5}{2}u^4 + 10u(\partial_s u)^3 + (\partial_s^2 u)^2.
\end{array}$$

$$\begin{aligned}
n=1: &\quad \partial_{t_1}u + \partial_s u = 0, \\
n=2: &\quad \partial_{t_2}u + 6u\partial_s u + \partial_s^3 u = 0, \\
n=3: &\quad \partial_{t_3}u + 30u^2\partial_s u + 20\partial_s u\partial_s^2 u + 10u\partial_s^3 u + \partial_s^5 u = 0.
\end{aligned}$$

3-15. Proposition. [AM]

(1) *ω is a cocycle 2-form.*

(2) *The KdVH flow has the Hamiltonian structures with their Hamiltonian,*

$$H_n := \int_{S^1} h_n\ ds, \quad (n \ge 0),$$

with involutive relations for the Poisson bracket, $\{H_n, H_m\} := \omega(X_n, X_m)$,

$$\{H_n, H_m\} = 0, \quad \textit{for all } n, m.$$

(3) *The n-th KdV flow has infinite conserved quantities H_m $n \in \mathbb{Z}_{\ge 0}$.*

(4) *We have the relation,*

$$[\partial_{t_n}, \partial_{t_m}]u = 0, \quad \textit{for all } n, m.$$

(5) *For an arbitrary curve γ, the n-th $(n \ge 1)$ KdV flow is uniquely determined.*

Proof. We will prove these following to the arguments in [AM]. First we will show that $i_X\omega$ is exact: For all $n > 0$, we have the relation,

$$i_{X_n}\omega(v) = \omega(X_n(u), v) = \int_{S^1} ds \frac{\bar{\delta} h_n}{\bar{\delta} u} v = (dH_n)(v), \quad \text{for } n \geq 1.$$

Hence $X_n(u)$ is a Hamiltonian vector field from the Definition 3-12 (2). Our system is a Hamiltonian system and the n-th KdV equation is given by,

$$u_{t_n} = X_n(u).$$

Next we will show that the KdVH flow is involutive. As the time t_m development of H_n is given by

$$\partial_{t_m} H_n = \int \frac{\bar{\delta} h_n}{\bar{\delta} u} \partial_{t_m} u = \int \{H_n, H_m\},$$

the involution relations are important. From the Definition 3-12, we have relations for $n \geq 1$,

$$\begin{aligned} X_n &= \partial_s \frac{\bar{\delta} h_n}{\bar{\delta} u} \\ &= \underline{\Omega} \frac{\bar{\delta} h_{n-1}}{\bar{\delta} u}. \end{aligned}$$

Since in terms of ω in Definition 3-13 (1), the Poisson bracket between H_n's are given by $\{H_n, H_m\} = \omega(X_n, X_m)$, we obtain the following relation for $n, m > 0$:

$$\begin{aligned} \{H_n, H_m\} &= \int_{S^1} ds \frac{\bar{\delta} h_n}{\bar{\delta} u} X_m(u) \\ &= \int_{S^1} ds \frac{\bar{\delta} h_n}{\bar{\delta} u} \underline{\Omega} \frac{\bar{\delta} h_{m-1}}{\bar{\delta} u} \\ &= \int_{S^1} ds \underline{\Omega} \frac{\bar{\delta} h_{n-1}}{\bar{\delta} u} \frac{\bar{\delta} h_m}{\bar{\delta} u} \\ &= \{H_{n+1}, H_{m-1}\}. \end{aligned}$$

Using this relations and noting $\{H_n, H_m\} = -\{H_m, H_n\}$, we will prove the involutive relation. When both n and m are even or both n and m are odd,

$$\{H_n, H_m\} = \{H_{(n+m)/2}, H_{(n+m)/2}\} = 0.$$

On the other hand, when n is odd and m is even,

$$\{H_n, H_m\} = \{H_{(n+m-1)/2}, H_{(n+m-1)/2+1}\} = \{H_{(n+m-1)/2+1}, H_{(n+m-1)/2}\} = 0.$$

Hence H_n's are involutive and the KdVH flow has infinite conserved quantities.

We can express the relation $\{H_n, H_m\} = 0$ by using a vector representation for $n, m > 0$,

$$[X_n, X_m] = 0.$$

In the solution of the KdV hierarchy, we can identify ∂_{t_n} with X_n itself: $\partial_{t_n} \equiv X_n$. Hence we obtain (4).

Further (5) can be proved as follows. For a given curve γ, we uniquely have the data, u,$\partial_s u$, $\partial_s^2 u$, $\cdots$. The KdV equations are given by

$$\partial_t u = f(u, \partial_s u, \partial_s^2 u, \cdots).$$

Hence for an arbitrary curve $\gamma \in \mathcal{M}^{\mathbb{P}}_{\text{elas}}$, the KdVH flow is uniquely determined by the KdV hierarchy. Due to the integrability, the "time" development of the γ is stably determined. ∎

Since the KdVH flow is a Hamiltonian system with infinite time parameters, we can find a group $g \in G$ such that $\gamma_{t+t'} = g_{t'}\gamma_t$. The multiplication is given as $g_{t'} g_t = g_{t'+t}$. g_0 is unit and g_{-t} is the inverse of g_t. Further Proposition 3-15 (4) means that $[\partial_{t_1}, \partial_{t_n}]u = 0$ and the projection of $\pi^{\mathbb{P}}_{\text{elas}} : \mathbb{M}^{\mathbb{P}}_{\text{elas}} \to \mathcal{M}^{\mathbb{P}}_{\text{elas}}$ consists with the KdV flow.

Further as solving the KdV hierarchy is an initial problem with the first derivative with respect to the time, for an arbitrary $\gamma \in \mathbb{M}^{\mathbb{P}}_{\text{elas}}$ we can find the KdVH flow to which γ belongs as an initial state.

We will give a proposition as a summary of the above arguments.

3-16. Proposition.

(1) *There is an Abelian group* $G := \{\exp(\sum_n t_n \partial_{t_n}) \mid t_n \in \mathcal{V}^\infty\}$ *acting on the moduli spaces* $\mathbb{M}^{\mathbb{P}}_{\text{elas}}$ *and* $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$*, whose orbits are identified with the KdVH flow.*

(2) *There is a fixed normal subgroup* G_0 *of* G, $G_0 = \{g_{t_1} \mid t_1 \in \mathbb{R}\} \approx \mathrm{U}(1)$; G_0 *trivially acts upon* $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$: $\gamma = g_{t_1}\gamma$ *for* $\gamma \in \mathcal{M}^{\mathbb{P}}_{\text{elas}}$ *and* $g_{t_1} \in G_0$.

(3) *The group* G/G_0 *acts on* $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$.

Hence Proposition 3-11 (2) is proved. We can express the equivalent class in $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$ by the group action in the following proposition.

3-17. Proposition.

(1) *Fixing* $\gamma \in \mathcal{M}^{\mathbb{P}}_{\text{elas}}$, G/G_0 *whose element is given as* $g_{t_2,t_3,\ldots}$ *transitively acts upon* $\mathfrak{C}[\gamma]$*: For an arbitrary* $\gamma' \in \mathfrak{C}[\gamma]$*, we can find an element* $g_{t_2,t_3,\ldots}$ *of the group* G/G_0 *such that* $\gamma = g_{t_2,t_3,\ldots}\gamma'$.

(2) *For an arbitrary* $\gamma \in \mathcal{M}^{\mathbb{P}}_{\text{elas}}$*, there exists the KdVH flow:* $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$ *can be decomposed,*

$$\mathcal{M}^{\mathbb{P}}_{\text{elas}} = \coprod \mathfrak{C}[\gamma].$$

(3) *For* $\gamma \in \mathbb{M}^{\mathbb{C}}_{\text{elas}}$*, the energy functional* $\mathcal{E}[\gamma]$ *is exactly conserved for the KdVH flow.*

Proof. (1) and (2) are obvious from the properties of group. (3) is proved because the energy $\mathcal{E}[\gamma]$ of the loop γ given by Definition 2-18 is identified with the conserved quantity of H_0. ∎

Hence Proposition 3-11 (7) is proved from Proposition 3-17 (3). By Propositions 3-11, 3-15, 3-16 and 3-17, we completely proved our main theorem 3-2.

As we have the classification of $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$, we will use it and go on to investigate the moduli space $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$ in rest of this paper because our purpose is to get some knowledge of the moduli space $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$.

For later convenience, we will introduce a quotient space. Due to Theorem 3-2 and Proposition 3-17, $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$ has natural projections induced by the equivalent relation $\underset{\text{KdVHf}}{\sim}$, *i.e.*, $\pi_{\text{KdVHf}} : \mathfrak{C}[\gamma] \mapsto (\gamma)$, where (γ) is a representative element of $\mathfrak{C}[\gamma]$.

3-18. Definition.

(1) We define a quotient space of the moduli space by, $\mathfrak{M}^{\mathbb{P}}_{\text{elas}} := \pi_{\text{KdVHf}} \mathcal{M}^{\mathbb{P}}_{\text{elas}} := \mathcal{M}^{\mathbb{P}}_{\text{elas}} / \underset{\text{KdVHf}}{\sim}$.
(2) The natural projection is denoted by $\pi_{\text{elas}} : \mathbb{M}^{\mathbb{P}}_{\text{elas}} \to \mathfrak{M}^{\mathbb{P}}_{\text{elas}}$.

3-19. Remark. We will comment on Proposition 3-15 (4),

$$[\partial_s, \partial_{t_n}] = 0 \quad \text{for } n > 0.$$

As the KdVH flow is very regular, we can regard $\mathfrak{C}[\gamma] \times S^1 \in \mathbb{M}^{\mathbb{P}}_{\text{elas}}$ as a manifold. Accordingly ∂_{t_n} are regarded as a vector field. We will use it as a generator of a cohomology in §7.

(1) It means the local length ds preserves for the KdVH flow.
(2) It can be interpreted as Frobenius integrability conditions.
(3) It is known as the compatibility condition or zero curvature conditions known in the soliton physics.

§4. Algebro-Geometric Properties of the KdV flow I –Algebraic Properties–

As we proved Theorem 3-4, we will use the relation between the moduli of a quantized elastica $\mathbb{M}^{\mathbb{P}}_{\text{elas}}$ and the KdV flow in order to give a finer classification, which is based on the study *finite type flow* in $\mathbb{M}^{\mathbb{P}}_{\text{elas}}$ and $\mathcal{M}^{\mathbb{P}}_{\text{elas}}$, in this section. However as this classification comes from the algebraic investigation of the KdV flow, we should replace the base function space in the category of the smooth functions by that of the formal power series in order to explain this classification, though we need some subtle treatments.

This section is devoted for investigations of a commutative differential ring, which were given by Mulase in [Mul], Burchnall and Chaundy in about seventy years ago [BC1, 2, Ba3], and Mumford in [Mum1]. Our argument basically follows the arguments of Mulase for the Schottky problem [Mul] and of Sato [SN, SS]. Following their theories, we will consider a part of the moduli of a quantized elastica using the formal power series. Since the part is dense in $\mathbb{M}^{\mathbb{P}}_{\text{elas}}$ as mentioned in Theorem 4-2, the replacement of the base field is not so critical.

Although investigation of γ as a real one-dimensional curve is our main subject, we deal with a hyperelliptic curve as a complex one-dimensional curve in the context of algebraic geometry in this section and next section. Thus readers should not confuse the terms “curve” in the categories of the differential geometry and the algebraic geometry. We basically refer the complex algebraic curve *algebraic curve*, *hyperelliptic curve* or *elliptic curve* whereas we call such a real curve just *curve*.

Let us start this section with the following lemma.

4-1. Lemma. *If there is a natural number N such that $\partial_{t_N} u$ is an eigen vector of the operator Ω with an eigenvalue $k \in \mathbb{C}$, i.e.,*

$$k\partial_{t_N} u = \Omega \partial_{t_N} u,$$

∂_{t_m} is a scalar multiplication of ∂_{t_N} for all $m \geq N$. Further by introducing t'_n $n > N$ and setting $\partial_{t'_n} := \partial_{t_n} - k^{n-N}\partial_{t_N}$, the relation becomes $\partial_{t'_n} u \equiv 0$.

Proof. This proof is easily from Definition 3-2 and Proposition 3-11 (5). ▮

Lemma 4-1 means that some orbits in infinite dimensional vector space $\mathcal{V}^\infty$ are essentially reduced to an orbit consisting of finite N dimensional vector space. Let us refer this flow *finite flow* or *finite N-type flow*.

Here we will give our second main theorem:

4-2. Theorem.

(1) *We will write the set of the finite type flow by $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas,finite}}$ and the set of finite g-type flow by $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}g}$. The moduli space of the elastica has decomposition,*

$$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas,finite}} := \coprod_{g<\infty} \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}g}.$$

(2) *$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas,finite}}$ is dense in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ with respect to the canonical topology determined by the KdVH flow.*

In order to prove Theorem 4-2, we prepare the knowledge of the KdV equation. A more concrete statement appears in Proposition 4-33. Before that, we will recall the result of the Whitney for the quasi-analytic system [Wh], which is easily proved by the Weierstrass preparation theorem.

4-3. Proposition. [Wh] *For a presheaf $\underline{\mathcal{C}^\infty}(\mathbb{R})$ of smooth functions over $\mathbb{R}$ and a presheaf $\underline{\mathcal{F}}(\mathbb{R})$ of $\mathbb{C}$-valued formal power series over $\mathbb{R}$, we have a surjective presheaf morphism,*

$$\eta : \underline{\mathcal{C}^\infty}(\mathbb{R}) \to \underline{\mathcal{F}}(\mathbb{R}),$$

i.e., for a germ $f \in \Gamma_p(\underline{\mathcal{C}^\infty}(\mathbb{R}))$ and $t_1 \in \mathbb{R}$ around t_1^0,

$$\eta(f) = \sum_i^\infty \left[\frac{d^i}{ds^i} f\right]_{t_1 = t_1^0} (t_1 - t_1^0)^i.$$

The map η is not injective, *e.g.*, due to a function f, $f(s) = 0$ at $s = s_0$ and $f(s) = \exp(-1/(s-s_0)^2)$ at otherwise points.

Since η is a local correspondence, the map η can be applied for the presheaves of the $\mathcal{C}^\infty$ functions and the formal power series over S^1, or $\eta : \underline{\mathcal{C}^\infty}(S^1) \to \underline{\mathcal{F}}(S^1)$.

On the other hand, for an arbitrary element in $\underline{\mathcal{C}^\infty}(S^1)$ we can find a sequence in the presheaf of the formal power series $\underline{\mathcal{F}}(S^1)$ which converges to the element using the same Weierstrass preparation theorem.

By using these properties, we will replace the base ring $\mathcal{C}^\infty(S^1, \mathbb{R})$ with the formal power series in this section.

In order to express the system of the KdV equation, we will mention the differential algebra and its division algebra over a commutative ring R. As we show in Definition 5-15 and Proposition 5-18, the hyperelliptic $\wp$ function obeys the KdV equation and has a singularity of the second order. Hence we might be ought to deal with Lourant expansion ring $\mathbb{C}[[t_1]][t_1^{-1}]$ as R. However as we are concerned with one of the KdVH flows which are finite and real valued, we deal only with a finite "real" valued part of $\wp$ and avoid the singular points. In other words, we employ a formal series ring $\mathbb{C}[[t_1]]$ as the ring R.

In this section, t_1 is dealt with as a generic parameter but can be regarded as a real (complex) number $t_1 \in \mathbb{R}$ ($\mathbb{C}$). After considering periodicity, we regard it as a point of $S^1 = \mathbb{R}/\mathbb{Z}$ in later. For a convenience, let $\partial_1 := \partial/\partial t_1$.

Here we assume that all algebra and subalgebra have unit as their definitions in this article.

4-4. Definition. [Mul, S, SN, SS]

(1) The differential ring $\mathfrak{D}^f$ is defined by

$$\mathfrak{D}^f := \{\sum_{k\geq 0}^{N} a_k \partial_1^k \mid N < \infty,\ a_k \in \mathbb{C}[[t_1]]\}.$$

(2) Let us identify commutative subalgebras B_1 and B_2 in $\mathfrak{D}^f$ if there exists an invertible element $r \in \mathbb{C}[[t_1]]^\times$ such that

$$B_1 = r B_2 r^{-1}.$$

We define *standard representation* B_s in the equivalent class $[B_1]$ by tuning $r \in \mathbb{C}[[t_1]]^\times$ such that it contains,

$$\partial_1^n + b_{n-2}\partial_1^{n-2} + \cdots + b_0 \in B_s.$$

(3) The degree of a differential operator and the projections $+$ and $-$ are defined by the same as the case $\mathfrak{E}^s$ in Definition 3-1.

(4) The micro-differential ring $\mathfrak{E}^f$ to $\mathfrak{D}^f$ is defined by,

$$\mathfrak{E}^f := \{\sum_{k=-\infty}^{N} a_k \partial_1^k \mid N < \infty,\ a_k \in \mathbb{C}[[t_1]]\}.$$

(5) The constant coefficient subring of $\mathfrak{E}^f$ is defined by,

$$\mathfrak{E}^c := \{\sum_{k=-\infty}^{N} a_k \partial_1^k \mid N < \infty,\ a_k \in \mathbb{C}\}.$$

(6) An invertible set $\mathfrak{W}^f$ in $\mathfrak{D}^f$ is defined by,

$$\mathfrak{W}^f := \{W \in \mathfrak{E}^f \mid W = 1 + \sum_{i=1}^{\infty} a_i \partial_1^{-i}, a_k \in \mathbb{C}[[t_1]]\}, \quad \mathfrak{W}^c := \mathfrak{W}^f \cap \mathfrak{E}^c.$$

For the readers who are not familiar with valuation, we will review it.

4-5. Definition. [Ha] $\mathcal{K}$ is a topological field. Let us call a topology space $\mathcal{E}$ *left linear topological space* if $\mathcal{E}$ satisfies

(1) $\mathcal{E}$ is a $\mathcal{K}$ linear space.
(2) A map from $\mathcal{K}\times\mathcal{E}$ to $\mathcal{E}$ $((\lambda, x)\mapsto \lambda x)$ and addition $x+y\in\mathcal{E}$ are continues.

4-6. Definition 4-6. [Ha]

(1) Let $\mathcal{K}$ be a field. A valuation of $\mathcal{K}$ with values in $\mathbb{Z}$ is a map val : $\mathcal{K}\to\mathbb{Z}$, for all $x, y\in\mathcal{K}$, $x, y\neq 0$,

$$\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y), \quad \mathrm{val}(x+y) \ge \min(\mathrm{val}(x), \mathrm{val}(y)).$$

and $\mathrm{val}(0) = \infty$.
(2) The set $\mathcal{R} := \{x\in\mathcal{K} \mid \mathrm{val}(x)\ge 0\ \}$ is a local subring, called *valuation ring.*
(3) The set $\mathfrak{m} := \{x\in\mathcal{K} \mid \mathrm{val}(x) > 0\ \}$ is called a *local ideal* of $\mathcal{R}$
(4) Let the metric of $\mathcal{K}$ be $|x| := \mathrm{e}^{-\mathrm{val}(x)}$ for $x\in\mathcal{K}$, which is called *non-Archimedian metric.*

For example, the valuation of a commutative ring $\mathbb{C}[x]$ is given by its degree, *i.e.*, for $f(x)\in\mathbb{C}[x]$, $\mathrm{val}(x) = \deg_x(x)$. For a more general commutative ring, we can find a local parameter by localization at a prime ideal and its valuation is given by its degree of the local parameter.

The valuation ring is a linear topological space due to the non-Archimedian metric [Ha]. Similarly, we have the following proposition [SN], which is naturally obtained.

4-7. Proposition. [SN]

(1) *When we define* $\mathfrak{E}^f{}_m := \{D\in\mathfrak{E}^f \mid \deg D\le m\ \}$, $\mathfrak{E}^f$ *has filter,*

$$\mathfrak{E}^f = \cup_m \mathfrak{E}^f{}_m, \quad \{0\} = \cap_m \mathfrak{E}^f{}_m, \quad \mathfrak{E}^f{}_m \subset \mathfrak{E}^f{}_{m+1}.$$

(2) $\mathfrak{E}^f$ *is a linear topological space with respect to this filter.*
(3) $\mathfrak{E}^f$ *is an infinite dimensional algebra given by the formal power sires whose element converges in the filter topology.*
(4) *In* $\mathfrak{E}^f$, *we can define valuations in* $\mathbb{C}[[t]]$ *and* $\mathfrak{E}^f$ *as* $P = \sum_{i=-\infty}^{\infty} a_i\partial_1^i \in \mathfrak{E}^f$

$$\mathrm{val}(a) := \max\{m\in\mathbb{N} | \partial_1^m a\neq 0\}, \quad \mathrm{val}(P) := \inf\{\mathrm{val}(a_i) - i\}.$$

Formally Proposition 4-7 is obvious from their definitions but we need rigorous arguments to justify them mathematically, which is written in [SW, SS, S2]. The differential operators appearing in the soliton theory and in the following arguments converge in this topology.

4-8. Lemma. [Mul, SN, SS]

(1) *The adjoint map for* $W\in\mathfrak{W}^f$, $Ad(W) : \mathfrak{E}^f \to \mathfrak{E}^f$ $(Ad(W)P = WPW^{-1})$, *defines the automorphism in* $\mathfrak{E}^f$. $Ad(W)|_{\mathfrak{E}^f{}_m}$ *is invariant, i.e.,* $Ad(W)\mathfrak{E}^f{}_m = \mathfrak{E}^f{}_m$
(2) *For an operator* $\tilde{L}\in\mathcal{L}$, *where*

$$\mathcal{L} := \{D\in\mathfrak{E}^f \mid D = \partial_s + \sum_{i=1}^{\infty} a_i\partial_s^{-i}\ \},$$

we can find a unique $W \in \mathfrak{W}^f$ modulo $\mathfrak{W}^c$ such that

$$Ad(W)\tilde{L} = \partial_s,$$

and then this relation induces the isomorphisms of

$$\mathfrak{W}^f/\mathfrak{W}^c \approx \mathcal{L}.$$

(3) *For every standard commutative subalgebra $\mathcal{A}^f \subset \mathfrak{D}^f$, there is a $\mathbb{C}$-subalgebra $\mathcal{A}^c \in \mathfrak{E}^c$ such that is $\mathbb{C}$-isomorphic to $\mathcal{A}^f$ and*

$$\mathcal{A}^c \cap \mathfrak{E}^c{}_- = \{0\}.$$

Proof. (1) is trivial. (2) Let us find $W = \sum_{i=0}^{\infty} w_i \partial_1^{-i}$ such that $\tilde{L} = W\partial_1 W^{-1}$. Noting $\tilde{L} = \partial_1 + \tilde{L}_-$, the relation is reduced to $[\partial_1, W] = -\tilde{L}_- W$, *i.e.*,

$$\partial_1 w_{k-1} = -\sum_{i+j+r=k, i\geq 2} \begin{pmatrix} 1-i \\ r \end{pmatrix} u_i \partial_1^r w_j.$$

Then we can recurrently determine w_i from small numbers since $\mathbb{C}[[t]]$ has indefinite integrals. When we find such W_1 and W_2, *i.e.*, $\tilde{L}W_a - W_a\partial_1 = 0$, then $W_1\partial_1 W_1^{-1}W_2 - W_1 W_1^{-1} W_2 \partial_1 = 0$ or $[\partial_1, W_1^{-1}W_2] = 0$. Hence $W_1^{-1}W_2 \in \mathfrak{W}^c$. (3): Let us take a monic element of $\mathcal{A}^f$ such that its form is

$$L_n = \partial_1^n + b_{n-2}\partial_1^{n-2} + \cdots + b_0.$$

Then we have $\tilde{L} = L_n \in \mathcal{L}$. Let $S \in \mathfrak{W}^f$ such that $\tilde{L} = S\partial_1 S^{-1}$. Then $\mathcal{A}^c := S^{-1}\mathcal{A}^f S$. For an arbitrary $P \in \mathcal{A}^f$, $S^{-1}PS$ belongs to $\mathfrak{E}^c$, *i.e.*, $[\partial_1, S^{-1}PS] = 0$ because $[\partial_1, S^{-1}PS] = S^{-1}[S\partial_1 S^{-1}, P]S = S^{-1}[L, P]S = 0$ due to the assumption $[P, L] = 0$. Further the inner automorphism preserves the order of the operator.∎

As we will not prove here, it is known that if we define a left $\mathfrak{E}^f$-module,

$$\mathfrak{V}^f := \mathfrak{E}^f/\mathfrak{E}^f t_1,$$

the homomorphism from $\mathfrak{E}^f$ to the endomorphism of $\mathfrak{V}^f$ is injective. In other words, the endomorphism is faithful if it can be regarded as a representation. $\mathfrak{V}^f$ has a valuational topology val and becomes a graded module. There the valuation and graded topology are identified. Further we have a natural $\mathfrak{E}^c$-module isomorphism [SN],

$$\mathfrak{V}^f \approx \mathfrak{E}^c.$$

Further we consider an embedding of a submodule $\mathfrak{V}^f{}_0$ into $\mathfrak{V}^f$ with zero-index map for a certain index, which can be regarded as a Grassmannian manifold in a certain sense. We note that the above isomorphism is not meaning of $\mathfrak{E}^f$-module.

In this article, we characterize such an embedding by a finite subset of natural numbers F, which can be regarded as the Weierstrass gap in the infinite point of a corresponding algebraic curve.

Further we should note that the adjoint map Ad is the key of the Sato theory and in this section, we sometimes call it *gauge* transformation.

4-9. Definition. [Mul] A $\mathbb{C}$-subalgebra $\mathcal{A}^c$ in $\mathfrak{E}^c$ is called a *rank one subalgebra* if it has $\mathbb{C}$-linear basis whose indices corresponds to all of integer except a finite subset F, *i.e.*,

$$\mathbb{N} - F = \mathbb{N}_{\mathcal{A}^c} := \{n \in \mathbb{N} \mid \exists P \in \mathcal{A}^c \text{ such that } \mathrm{ord}(P) = n\ \}$$

and

$$\mathcal{A}^c \cap \mathfrak{E}^c{}_- = \{0\}.$$

As $\mathcal{A}^c$ is a $\mathbb{C}$-algebra, there is a monic element P_n in $\mathcal{A}^c$ of order $n \in \mathbb{N} - F$ with $P_0 := 1$. Then $\{P_n | n \in \mathbb{N} - F\}$ forms a $\mathbb{C}$-linear basis of $\mathcal{A}^c$. In other words arbitrary $P \in \mathcal{A}^c$ can be represented by $\mathbb{C}$-linear combinations of monic P_n elements. In fact if the order of P is m, there exists $c \in \mathbb{C}$ such that the order of $P - cP_m \in \mathcal{A}^c$ must be less than m. Such a recursion process gives us the representations.

4-10. Lemma. [Mul]

Let $\mathcal{A}^c \neq \mathbb{C}$ be a rank one subalgebra, and P and Q be elements in $\mathcal{A}^c$ whose orders are coprime.

(1) $\dim_{\mathbb{C}}(\mathcal{A}^c/\mathbb{C}[P,Q]) < +\infty$.

(2) *$\mathcal{A}^c$ is finite $\mathbb{C}[P]$-module. There is a nontrivial polynomial $f(x,y) \in \mathbb{C}[x,y]$ such that $f(P,Q) = 0$.*

(3) *The transcendence degree of $\mathcal{A}^c$ over $\mathbb{C}$ is one.*

(4) *By regarding $\mathcal{A}^c$ as a graded module with respect to degree of differential operators:*

$$\mathcal{A}^{c(n)} := \{P \in \mathcal{A}^c \mid \mathit{ord}(P) \le n\ \},$$
$$\mathcal{A}^c{}_n := \mathcal{A}^{c(n)} \oplus \mathcal{A}^{c(n-1)} \cdot I \oplus \mathcal{A}^{c(n-2)} \cdot I^2 \oplus \cdots \oplus \mathcal{A}^{c(0)} \cdot I^n,$$
$$\mathrm{gr}\mathcal{A}^c = \oplus_{n=0}^{\infty} \mathcal{A}^c{}_n,$$

we regard $\mathrm{Proj}(\mathrm{gr}\mathcal{A}^c)$ *as an algebraic curve C. Here I is the identity of $\mathcal{A}^c$.*

(5) *Let* $\mathrm{H}^1(\mathcal{A}^c) = \mathfrak{E}^c/\mathcal{A}^c \oplus \mathfrak{E}^c{}_-$. *We have*

$$\mathrm{H}^1(C, \mathcal{O}_C^\times) = \mathrm{H}^1(\mathcal{A}^c),$$

where $\mathcal{O}$ is the sheaf of holomorphic functions on C of (4) and $\mathcal{O}^\times$ is a multiplicative subset of $\mathcal{O}$.

Proof. Let $\mathrm{GCD}(m,n)$ denote the greatest common divisor of two non-negative integer m and n. Since the rank of $\mathcal{A}^c$ is unit, we have the relations

$$1 = \min\{\mathrm{GCD}(\mathrm{ord}(P'), \mathrm{ord}(Q')) \mid P', Q' \in \mathcal{A}^c\ \}.$$

and the orders of P and Q are coprime. Hence $\mathbb{C}[P,Q] \subset \mathcal{A}^c$. As $N_{\mathbb{C}[P,Q]} = \mathbb{N}$ and $\mathbb{C}[P,Q]$ is $\mathbb{C}$-linear vector space, $N_{\mathcal{A}^c} - N_{\mathbb{C}[P,Q]}$ must be finite set. Hence $\dim_{\mathbb{C}}(\mathcal{A}^c/\mathbb{C}[P,Q])$ must finite. On the other hand, since $\mathbb{N} - \{\mathrm{ord}\{P^m, Q^n\} \mid m, n \in \mathbb{Z}_{\ge 0}\}$ must be finite set, P and Q satisfy an algebraic relation $f(P,Q) = 0$. Further the proofs of (4) and (5) are due to theory of an ordinary commutative ring [Ha]. ∎

We note that F in Definition 4-9 is related to the Weierstrass gap at infinity point of the algebraic curve C.

After this point, we will concentrate our attention only on the operator $L = \partial_1^2 + u$, which is related to the KdV equation:

$$\mathcal{L}_2 := \{D \in \mathfrak{E}^f \mid D = \partial_1^2 + u, \quad u \in \mathbb{C}[[t_1]]\}.$$

We give its related operators as examples.

4-11. Example.

$$
\begin{aligned}
L^{1/2} &= \partial_1 + \frac{1}{2}u\partial_1^{-1} - \frac{1}{4}(\partial_1 u)\partial_1^{-2} + \frac{1}{8}((\partial_1^2 u) - u^2)\partial_1^{-3} \\
&\quad + \frac{1}{16}(6u(\partial_1 u) - \partial_1^3 u)\partial_1^{-4} - \frac{1}{32}(-2u^3 + 14u(\partial_1^2 u) + 11(\partial_1 u)^2 - (\partial_1^3 u))\partial_1^{-5} + \cdots, \\
4L^{3/2} &= 4\partial_1^3 + 3\partial_1 u + 3u\partial_1 + \left(\frac{1}{2}\partial_1^2 u + \frac{3}{2}u^2\right)\partial_1^{-1} + \cdots ., \\
16L^{5/2} &= 16\partial_1^5 + 40u\partial_1^3 + 60(\partial_1 u)\partial_1^2 + 50(\partial_1^2 u)\partial_1 + 30u^2\partial_1 15(\partial_1^3 u) + 30u(\partial_1 u) \\
&\quad + \left(5\left(u^3 + \frac{1}{2}(\partial_1 u)^2\right) + \partial_1 f(u, \partial_1 u, \cdots)\right)\partial_1^{-1} + \cdots .
\end{aligned}
$$

Here $\partial_1 f(u, \partial_1 u, \cdots)$ is a functional of $u, \partial_1 u, \cdots$.

Let us fix the operator $P = \partial_1^2$ of $\mathcal{A}^c$ in Lemma 4-10 because we only consider $L = W\partial_1^2 W^{-1}$. From the primitive number theory, for an odd number m and an integer $n(> m)$, we find $a, b \in \mathbb{Z}_{\geq 0}$ such that

$$
n = am + 2b, \quad (a, b \in \mathbb{Z}_{\geq 0}). \tag{4.1}
$$

When we fixed $\mathcal{A}^c$ as a rank one subalgebra, the partner Q of $P \equiv \partial_1^2$ in the Lemma 4-10 is an operator whose order is given by an odd number $2g+1$. Thus F in Definition 4-9 is given by a smaller sequence of odd numbers, $\{1, 3, 5, 7, 9, \cdots, 2g-1\}$. Let us introduce a set of such subrings $\mathcal{A}^c$ in $\mathfrak{E}^c$.

4-12. Definition.

$$
\begin{aligned}
\mathfrak{A}^c := \{\mathcal{A}^c \mid\ & \mathcal{A}^c \text{ is a rank one subalgebra}, \\
& \exists W \in \mathfrak{W}^f \ \text{ such that } W\mathcal{A}^c W^{-1} \in \mathfrak{D}^f \text{ is a commutative subalgebra}, \\
& \mathbb{N} - \mathbb{N}_{\mathcal{A}^c} \subset \{1, 3, \cdots, 2g-1\},\ g < \infty\ \}.
\end{aligned}
$$

For the case of $g = 1$, $\mathcal{A}^c \equiv W^{-1}\mathbb{C}[L, [L^{1/2}]_+]W$. Since $[L^{1/2}]_+ \equiv \partial_1$, $[L, \partial_1] = 0$ and thus u must be $\mathbb{C}$. In other words, Q must be ∂_1, $\mathbb{C}[\partial_1^2, \partial_1] \equiv \mathbb{C}[\partial_1]$. For the case $g = 1$, it becomes an ordinary polynomial ring.

4-13. We recall that an algebraic curve with a morphism to $\mathbb{P}$ of order two is called hyperelliptic curve. A hyperelliptic curve C_g of genus g ($g \geq 1$), including the case of elliptic curve, is given by the homogeneous equation,

$$
Y^2 Z^{2g-1} = h_g(X, W) := \lambda_0 Z^{2g+1} + \lambda_1 X Z^{2g} + \lambda_2 X^2 Z^{2g-1} + \cdots + \lambda_{2g+1} X^{2g+1},
$$

where $\lambda_{2g+1} \equiv 1$ and λ_j's are complex values.

4-14. Lemma. *Let $L = \partial_1^2 + u \equiv W\partial_1^2 W^{-1} \in \mathcal{L}_2$ to $W \in \mathfrak{W}^f$.*

(1) $L^{n/2} = W\partial_1^n W^{-1}$.
(2) $[L^{2n}{}_+, L] \equiv [L^{2n}, L] \equiv 0$.
(3) *The set of the differential operators in $\mathfrak{D}^f$ which commute with a given operator $L_2 \in \mathfrak{D}^f$ is itself a commutative subalgebra of $\mathfrak{D}^f$.*

(4) *$\mathcal{A}^c \in \mathfrak{A}^c$ is $\mathbb{C}[\partial_1^2]$-module and by considering*

$$\mathfrak{A}^c = \sum \mathcal{A}^c,$$

$\mathfrak{A}^c$ is also $\mathbb{C}[\partial_1^2]$-module.

(5) *For an arbitrary $\mathcal{A}^c \in \mathfrak{A}^c$, we can find $Q_g \in \mathcal{A}^c$ which satisfies an affine equation,*

$$Q_g^2 = h_g(\partial_1^2, 1),$$

so that there is a $W \in \mathfrak{W}^f/\mathfrak{W}^c$ such that $W\partial_1^2 W^{-1} = L$ and WQW^{-1} are commutative in $\mathfrak{D}^f$. Further we have found a hyperelliptic curve $C = \mathrm{Proj}(\mathrm{gr}\mathcal{A}^c)$ and

$$\mathrm{H}^1(C, \mathcal{O}_C) = \mathrm{H}^1(\mathcal{A}^c),$$

which are generated by $< \partial_1, \partial_1^3, \cdots, \partial_1^{2g-1} >$.

Proof. (1) and (2) can be shown by direct computations. On (3), we consider a commutative differential ring in $\mathfrak{D}^f$ such that $\mathcal{B} := \{P \in \mathfrak{D}^f \mid [P, L] = 0 \}$. Since $L^{1/2} = W\partial_1 W^{-1}$, $[\partial_1, W^{-1}PW] = W^{-1}[L^{1/2}, P]W = 0$ because of the assumption. Hence $W^{-1}PW$ is an element of $\mathfrak{E}^c$ and thus we can find $\mathcal{A}^c \in \mathfrak{A}^c$ such that $\mathcal{B} = W^{-1}\mathcal{A}^c W$. Hence $\mathcal{B}$ is a commutative ring. Next (4) is trivial. From the definition Lemma 4-10, and Eq. (4.1), we reach (5). ∎

Next we will consider the filter structure in $\mathfrak{A}^c$ and its completion with respect to the filtration.

4-15. Proposition. *Let us define a filter,*

$$F_g\mathfrak{A}^c := \{\mathcal{A}^c \in \mathfrak{A}^c \mid \mathbb{N} - \mathbb{N}_{\mathcal{A}^c} \subset \{1, 3, \cdots, 2g-1\}\}.$$

This satisfies the following relations:

(1)

$$F_g\mathfrak{A}^c \subset F_{g+1}\mathfrak{A}^c, \quad F_n\mathfrak{A}^c \equiv 0, \quad n < 0.$$

(2) *By letting ${\mathfrak{A}^c}_g := F_g\mathfrak{A}^c/F_{g-1}\mathfrak{A}^c$, there is a large gauge transformation between ${\mathcal{A}^c}_1, {\mathcal{A}^c}_2 \in {\mathfrak{A}^c}_g$, i.e., there exists $W \in \mathfrak{W}^f$ such that ${\mathcal{A}^c}_1 = W{\mathcal{A}^c}_2 W^{-1}$.*

(3) *The direct limit of the filtration gives*

$$\begin{aligned}\overline{\mathfrak{A}^c} &:= \lim_{\to} F_g\mathfrak{A}^c \\ &= \{\mathcal{A}^c \in \mathfrak{E}^c \mid \exists W \in \mathfrak{W}^f \text{ such that} \\ &\quad W\mathcal{A}^c W^{-1} \in \mathfrak{D}^f \text{ is a subalgebra}, \mathbb{N} - \mathbb{N}_{\mathcal{A}^c} \subset 2\mathbb{N} - 1 \}.\end{aligned}$$

Proof. (1) and (3) are obvious. (2) is due to the proof of Proposition 4-16 (2). ∎

For each element $\mathcal{A}^c \in {\mathfrak{A}^c}_g$, we consider the correspondence in Lemma 4-10 (4), *i.e.*, $\mathrm{Proj}(\mathrm{gr}{\mathcal{A}^c}_g)$. It turns out that ${\mathfrak{A}^c}_g$ is isomorphic to the set of the hyperelliptic curves with genus g.

4-16. Proposition.

(1) *The set* $\mathfrak{A}^f$ *of commutative subrings in* $\mathfrak{D}^f$ *inherits from the above filtration of* $\mathfrak{A}^c$.

(2) *For any elements* L_1 *and* L_2 *in* $\mathcal{L}_2$, *there is a gauge transformation* $W \in \mathfrak{W}^t/\mathfrak{W}^c$ *such that*

$$L_1 = W L_2 W^{-1}.$$

Proof. (1) is trivial. (2): There is an element $W_a \in \mathfrak{W}^t/\mathfrak{W}^c$ such that $L_a = W_a \partial_1^2 W_a^{-1}$ for $(a = 1, 2)$. Hence $L_2 = W_2 W_1^{-1} L_1 W_1 W_2^{-1}$. ∎

As we described the tools and their properties for the differential ring over $\mathbb{C}[[t_1]]$, we will extend its base field to $\mathbb{C}[[t_1, t_2, \cdots]]$. However before we will give the extension in Definition 4-19, we digress and show a connection between Ω in Definition 3-2 and L in $\mathcal{L}_2$. Following the arguments in [D], we firstly prepare a lemma.

4-17. Lemma. [D] *The "resolvent" operator for* $L = \partial_1^2 + u$,

$$T^{(\pm)} := \left[\frac{1}{2z^2} \sum_{r=-\infty}^{\infty} (\pm z)^r L^{-r/2} \right]_- ,$$

has the following properties:

(1) $(T^{(+)} + T^{(-)}) = (L - z^2)^{-1}$.

(2) $[T^{(\pm)}(L - z^2)]_- = [(L - z^2)T^{(\pm)}]_- = 0$.

(3) *When we define a map for a* $X \in \mathfrak{E}^s$, *called Adler map [D],*

$$\mathfrak{h}(X) := [(L - z^2)X]_+(L - z^2) - (L - z^2)[X(L - z^2)]_+,$$

we have the relation, $\mathfrak{h}(T^{(\pm)}) = 0$.

(4) $T^{(\pm)}$ *has a formal expansion,*

$$T^{(\pm)} = \sum_{r=1}^{\infty} S_r^{(\pm)} \partial_1^{-r}.$$

Proof. (1) is trivial. (2) is given by the relation,

$$\begin{aligned}
[(L - z^2)T^{(\pm)}]_- &= \frac{1}{2} \left[(L - z^2) \left[\sum_{r=-\infty}^{\infty} (\pm z)^r L^{-r/2} \right]_- \right]_- \\
&= \frac{1}{2} \left[(L - z^2) \sum_{r=-\infty}^{\infty} (\pm z)^r L^{-r/2} \right]_- \\
&= \frac{1}{2} \left[\sum_{r=-\infty}^{\infty} \left((\pm z)^r L - z^2 (\pm z)^r \right) L^{-r/2} \right]_- .
\end{aligned}$$

It is clear that it vanishes. (3) is proved due to the property of the Adler map,

$$\mathfrak{h}(X) \equiv -[(L - z^2)X]_-(L - z^2) + (L - z^2)[X(L - z^2)]_-.$$

(4) is obvious from the definition of the resolvent. ∎

Due to the lemma, we gave the connection.

4-18. Proposition. [D]

(1)

$$[2^{(n-1)}[L^{(2n-1)/2}]_+, L] = \Omega_1^n \partial_1 u,$$

where $\Omega_1 := \partial_1^2 + 2u + 2\partial_1 u \partial_1^{-1}$.

(2) *By letting* $\overline{h}_n = \mathrm{res}\dfrac{2^{2n}}{2n+1}L^{(2n-1)/2}$, *("res" means the coefficient of* ∂_1^{-1}*), we have*

$$\partial_1 \overline{h}_n = \Omega_1 \partial_1 \overline{h}_{n-1}.$$

Proof. Due to the condition $\mathfrak{h}(T^{(\pm)}) = 0$, we can determine the first two coefficients S_1 and S_2 as

$$\partial_s^3 S_1^{(\pm)} + 2(\partial_s u)S_1^{(\pm)} + 4(u + z^2)\partial_s S_1^{(\pm)} = 0,$$

$$\partial_s^2 S_2^{(\pm)} = -\frac{1}{2}\partial_s S_1^{(\pm)}.$$

Let us consider the following operator,

$$(T^{(+)} - T^{(-)}) = \left[\sum_{r=-\infty}^{\infty} z^{2r+1}L^{-(2r+1)/2}\right]_- .$$

The left hand side in the relation,

$$[L^{(2r+1)/2}]_+ L - L[L^{(2r+1)/2}]_+ = L[L^{(2r+1)/2}]_- - [L^{(2r+1)/2}]_- L,$$

appears as a coefficient of z^{2r-1} in the series $\mathfrak{h}(T^{(+)} - T^{(-)})$ with respect to z. Thus we are concerned with $S_r := S_r^{(+)} - S_r^{(-)}$, which must have the expansion,

$$S_r = \sum_{i=-\infty}^{\infty} S_r^{(i)} z^{2i+1}.$$

Comparing the coefficients in z^{2r-1}, we obtain,

$$4\partial_1 S_1^{(i+1)} = (\partial_s^3 + 2(\partial_s u) + 4u\partial_1)S_1^{(i)} = \Omega_1 \partial_1 S_1^{(i)}.$$

We have the relation,

$$[[L^{(2r+1)/2}]_+, L] = \frac{1}{4}\partial_1 S_1^{2r+1} = \frac{1}{4^r}\Omega_1^r \partial_1 S_1^{(1)},$$

with $S_1^1 = -u/2$. Then we let $\overline{h}_n$ identified with S_n by tuning its coefficient. ∎

As we finished the digression, we extend $\mathbb{C}[[t_1]]$ to $\mathbb{C}[[t_1, t_2, \cdots]]$. In the extension of the valuation over $\mathbb{C}[[t_1]]$ to that of $\mathbb{C}[[t_1, t_2, \cdots]]$, let the degree of t_i^n be $(2i-1)n$.

4-19. Definition. [Mul, SN]

(1) The differential ring $\mathfrak{D}^t$, and its related set and ring are defined by,

$$\mathfrak{D}^t := \{\sum_{k\ge 0}^{N} a_k \partial_1^k \mid N<\infty,\ a_k \in \mathbb{C}[[t_1,t_2,\cdots]]\},$$

$$\mathfrak{E}^t := \{\sum_{k=-\infty}^{N} a_k \partial_1^k \mid N<\infty,\ a_k \in \mathbb{C}[[t_1,t_2,\cdots]]\}, \quad \mathfrak{E}^t = \mathfrak{D}^t + \mathfrak{E}^t{}_-,$$

$$\mathcal{L}_2^t := \{D\in \mathfrak{E}^t \mid D = \partial_1^2 + u, \quad u \in \mathbb{C}[[t_1,t_2,t_3,\cdots]]\}.$$

(2) By letting $\mathrm{val}(t_i^n) := (2i-1)n$, we extend the valuation of $\mathfrak{D}^t$ and $\mathfrak{E}^t$, which are also called valuations of $\mathfrak{D}^t$ and $\mathfrak{E}^t$.

(3)

$$\mathfrak{W}^t := \{W \in \mathfrak{E}^t \mid W = 1 + \sum_{i=1}^{\infty} w_i \partial_1^{-i}\ \}.$$

(4)

$$\hat{\mathfrak{D}}^t := \{\ P = \sum_{i=0}^{\infty} a_i\partial_1^i \in \mathfrak{D}^t \mid \exists N \in \mathbb{N},\ \mathrm{val}(a_i) > i-N \ \text{ for } \forall i \gg 0\ \},$$

$$\hat{\mathfrak{E}}^t := \{\ P = \sum_{i=-\infty}^{\infty} a_i\partial_1^i \in \mathfrak{E}^t \mid \exists N \in \mathbb{N},\ \mathrm{val}(a_i) > i-N \ \text{ for } \forall i \gg 0\ \}.$$

We note that $\mathfrak{D}^t$, $\mathfrak{E}^t$ and so on, have natural embeddings of $\mathfrak{D}^f$, $\mathfrak{E}^f$ and so on, *e.g.*,

$$\mathfrak{D}^f \ni P(t_1) \mapsto P(t_1, 0, 0, \cdots) \in \mathfrak{D}^t.$$

Using the embeddings, we regard $\mathfrak{D}^f$ as a subring of $\mathfrak{D}^t$ as following.

4-20. Definition.

(1) *The moduli space of the KdV equations is defined by*

$$\mathbb{M}_{\mathrm{KdV}} := \{u \in \mathbb{C}[[t_1,t_2,\cdots]] \mid \partial_{t_n} u - \Omega_1^{n-1}\partial_1 u = 0 \textit{ for } \ \forall n\ \}, \quad \mathcal{M}_{\mathrm{KdV}} := \mathbb{M}_{\mathrm{KdV}}/(t_1),$$

where $\Omega_1 := \partial_1^2 + 2u + 2\partial_1 u \partial_1^{-1}$. *Here* y *is an element of the vector space generated by* $t_1, t_2, \cdots$.

(2) *If* $\tilde{L} \in \mathcal{L}^t := \{D \in \mathfrak{E}^t \mid D = \partial_s + \sum_{i=1}^{\infty} a_i \partial_s^{-i}\ \}$ *and* $P \in \mathfrak{D}^t$ *satisfy* $[P, \tilde{L}] \in \mathfrak{E}^t{}_-$, *the equation* $[\partial_y - P, \tilde{L}] = 0$ *is called Lax equation and* (P, L) *is Lax pair.*

4-21. Proposition. [Mul] *Let* $L := \partial_1^2 - u \in \mathcal{L}_2^t$.

(1) $[\partial_{t_n} - 2^{2(n-1)}[L^{(2n-1)/2}]_+, L] = 0$ *is the Lax equation.*

(2) *For an arbitrary* $P \in \mathfrak{D}^t$ *of the Lax pair* (P, L), P *can be expressed by*

$$P = \sum_{j=1}^{n} c_j [L^{j/2}]_+,$$

where $c_j \in \mathbb{C}[[t_2, t_3, \cdots]]$.

(3) *If and only if* u *satisfies* $[\partial_y - P, L^{1/2}] = 0$, $[\partial_y - P, L] = 0$.

(4) *The equation* $[\partial_{t_n} - 2^{2(n-1)}[L^{(2n-1)/2}]_+, L] = 0$ *gives the* n*-th KdV equation,* $\partial_{t_n} u - \Omega_1^{n-1}\partial_1 u = 0$, *and thus we have a bijection*

$$\mathbb{M}_{\mathrm{KdV}} \approx \{L \in \mathcal{L}_2^t \mid [\partial_{t_n} - 2^{2(n-1)}[L^{(2n-1)/2}]_+, L] = 0,\ n > 1\ \}.$$

Here $\approx$ *is given by the correspondence between* u *and* $L = \partial_1^2 + u$.

Proof. First we consider (3). Let $L = W\partial_1^2 W^{-1}$. $[\partial_y - P, W\partial_1 W^{-1}] = 0$ gives $[W(\partial_y - P)W^{-1}, \partial_1] = 0$ and then we obtain $[W(\partial_y - P)W^{-1}, \partial_1^2] = 0$ and $[\partial_y - P, L] = 0$. For an operator $Q \in \mathcal{A}^c$, $[Q, \partial_1^2] = 0$ means $(\partial_1^2 Q) + 2(\partial_1 Q)\partial_1 = 0$, *i.e.*, $(\partial_1^2 Q) = 0$ and $(\partial_1 Q) = 0$. Hence $[W(\partial_y - P)W^{-1}, \partial_1^2] = 0$ also means $[\partial_y - P, L^{1/2}] = 0$. (1) It is know that $[[L^{j/2}]_+, L^{1/2}] \in \mathfrak{E}^t{}_-$. Due to (3), (1) is proved. Next we consider (2). $[L^{j/2}]_+$ is a monic operator. Hence if order of P is n, there exists $c \in \mathbb{C}$ such that the order of $P - c[L^{n/2}]_+ \in \mathfrak{D}^t$ is $n-1$. By induction, we have the results in (2). Proposition 4-18 (1) leads us to (4). ∎

Here we will translate the relations in terms of geometrical language. Due to Proposition 4-21 (4), we also denote the right hand side there by $\mathbb{M}_{\mathrm{KdV}}$.

4-22. Lemma. [Mul, SN, SS] *Let* $L := \partial_1^2 - u = W^{-1}\partial_1^2 W \in \mathcal{L}_2^t$,

$$dL := \partial_1 L dt_1 + \partial_2 L dt_2 + \partial_3 L dt_3 + \cdots,$$

$$dW := \partial_1 W dt_1 + \partial_2 W dt_2 + \partial_3 W dt_3 + \cdots,$$

$$dZ := 2L^{1/2}dt_1 + 4L^{3/2}dt_2 + 8L^{5/2}dt_3 + \cdots,$$

$$dZ_+ := 2[L^{1/2}]_+ dt_1 + 4[L^{3/2}]_+ dt_2 + 8[L^{5/2}]_+ dt_3 + \cdots, \quad Z = Z_+ + Z_-.$$

(1) *The Lax equation becomes*

$$dL = [Z_+, L], \quad dL = -[Z_-, L].$$

(2)

$$dZ_+ = \frac{1}{2}[Z_+, Z_+], \quad dZ_- = -\frac{1}{2}[Z_-, Z_-].$$

(3) $dL = [dW \cdot W^{-1}, L]$.

(4) $W^{-1}dW - Z_+ \in \mathfrak{D}^c dt_1 + \mathfrak{D}^c dt_2 + \mathfrak{D}^c dt_3 + \cdots$ *or by using the gauge freedom,*

$$dW = Z_+ W, \quad dW = -Z_- W.$$

Proof. (1) is trivial. (2): Noting $d^2 L \equiv 0$, $[L, dZ_+ - Z_+ Z_+] \equiv 0$ and then we obtain (2). (3): From $d(WW^{-1}) \equiv 0$, $dW^{-1} = -W^{-1}dWW^{-1}$. Hence $dL = d(W\partial_1^2 W^{-1})$ becomes the right hand side. (4): Using (2) and (3), $[dWW^{-1} - Z_+, L] = 0$, and we obtain $[W^{-1}dW - W^{-1}Z_+ W, \partial_1] = 0$. It implies (4). ∎

Here we note that the conditions $dZ_+ = [Z_+, Z_+]$ and so on are the Frobenius integrability conditions. Due to the conditions, the orbit as a dynamical system can be uniquely determined. Conclusively we have the following proposition on the orbit of the KdV equations. As its proof is a little bit complicate, we will give only the result.

4-23. Proposition. [Mul] *For* $L(0) = S(0)\partial^2 S(0)^{-1}$,

$$U(t) = exp(t_1\partial_1 + t_2\partial_1^3 + t_3\partial_1^5 + \cdots)S(0)^{-1} \in \hat{\mathfrak{E}}^t,$$

$U(t) = S(t)^{-1}Y$ *for* $S(t) \in \mathfrak{G}$ *and* $Y \in \hat{\mathfrak{D}}^t$, *we have the time development,*

$$L(t) = S(t)\partial_1^2 S(t)^{-1}.$$

4-24. Definition. [Mul]

(1) For $L \in \mathcal{L}_2^t$, if the map

$$T_0(R_{t,n}) \ni \frac{\partial}{\partial y} \mapsto \left.\frac{\partial L^{1/2}}{\partial y}\right|_{y=0} \in \mathfrak{E}^t{}_-$$

is injective, we say that $R_{t,n}$ is *effective*. Here we write $R_{t,n}$ as the orbit space generated by $t_1, t_2, \cdots, t_n$ and $T_0(R_{t,n})$ as its tangent space at the origin $0 \in R_{t,n}$.

(2) If for $L \in \mathcal{L}_2^t$ $R_{t,n}$ is not effective for $n > g$ but $n \leq g$ is effective, we say that $L = \partial_1^2 + u$ or u is *finite g type solution* of the KdV equation.

4-25. Lemma. [Mul, S, SN]

(1) *If there is a natural number* N *such that* $\partial_{t_N} u$ *is an eigen vector of the operator* Ω *with an eigenvalue* $k \in \mathbb{C}$, *i.e.,*

$$k\partial_{t_N} u = \Omega\partial_{t_N} u,$$

∂_{t_m} *is scalar multiplication of* ∂_{t_N} *for* $m \geq N$. *If not, we refer that* t_m *is effective.*

(2) *For the finite* g *solution of* L *and for* $n > g$, *we have the commutation relation,*

$$[2^{2(g+1)}[L^{(2g+1)/2}]_+, L] \equiv 0,$$

by construction t_n *in terms of a linear combination in* $\mathbb{C} < t_1, t_2, \cdots, t_g >$,

(3) *Let* $L \in \mathcal{L}_2^t$ *such that* $[2^{2(g+1)}[L^{(2g+1)/2}]_+, L] \equiv 0$ *and*

$$[\partial_{t_j} - 2^{2(j-1)}[L^{(2j-1)/2}]_+, L] = 0, \quad \textit{for } j < g,$$

is effective. Then we have a commutative subring $\mathcal{A}^t := \mathbb{C}[L, [L^{(2g+1)/2}]_+] \subset \mathfrak{D}^t$ *such that* $W \in \mathfrak{W}^t$, $\mathcal{A}^c = W^{-1}\mathcal{A}^t W \in \mathfrak{A}^c$, *and an isomorphism as* $\mathbb{C}$*-vector space,*

$$\mathrm{H}^1(\mathcal{A}^c) \approx \mathbb{C} < dt_1, dt_2, \cdots, dt_g > .$$

Proof. (1) is essentially the same as Lemma 4-1 and (2) is obvious from (1). So we will concentrate our attention on (3). The integrability conditions makes the conditions in $\mathfrak{D}^t$ reduced to those in $\mathfrak{D}^f$ as an initial state. Due to Lemma 4-14 (5), (3) is proved. ∎

4-26. Definition.

(1) The filter with respect to the effective differential equations is defined by

$$\begin{aligned} F_g\mathbb{M}_{\mathrm{KdV}} &:= \{L \in \mathcal{L}_2^t \mid [\partial_n - 2^{2(n-1)}[L^{(2n-1)/2}]_+, L] = 0 \text{ is not effctive for } n > g\} \\ &= \{L \in \mathcal{L}_2^t \mid [[L^{(2g-1)/2}]_+, L] \equiv 0\ \} \end{aligned}$$

and

$$F_g\mathbb{M}_{\mathrm{KdV}} \subset F_{g+1}\mathbb{M}_{\mathrm{KdV}}, \quad F_n\mathbb{M}_{\mathrm{KdV}} = \emptyset, \text{ for } n < 0.$$

(2) A set of finite g type solutions of the KdV equation is denoted by

$$\mathbb{M}_{\mathrm{KdV}g} := F_g\mathbb{M}_{\mathrm{KdV}} \setminus F_{g-1}\mathbb{M}_{\mathrm{KdV}}.$$

Due to Lemma 4-25 (3), $F_g\mathbb{M}_{\mathrm{KdV}}$ corresponds to $F_g\mathfrak{A}^c$ and the correspondence becomes a bijection by considering their appropriate quotient spaces.

As the system of the KdV equations is a dynamical system, there is a g-dimensional orbit in each solution space in $\mathbb{M}_{\mathrm{KdV}g}$ by neglecting its periodicity. We can regard it as a fiber bundle,

$$\begin{array}{ccc} \text{orbit} & \longrightarrow & \mathbb{M}_{\mathrm{KdV}g} \\ & & \big\downarrow \pi^g_{\mathrm{KdV}} \\ & & \mathfrak{M}_{\mathrm{KdV}g}. \end{array}$$

For each orbit space ${\pi^g_{\mathrm{KdV}}}^{-1}(p)$ at a point p in $\mathfrak{M}_{\mathrm{KdV}g}$, there is a commutative ring $\mathcal{A}^c \in {\mathfrak{A}^c}_g$ such that

$$T^*{\pi^g_{\mathrm{KdV}}}^{-1}(p) \approx \mathrm{H}^1(\mathcal{A}^c).$$

For later convenience, we also define a space $\mathfrak{M}_{\mathrm{KdVfinite}} := \coprod_g \mathfrak{M}_{\mathrm{KdV}g}$.

Next we will consider $\mathbb{M}_{\mathrm{KdV}}$ itself. $(F_g, \mathfrak{A}^c)$ has direct limit due to Proposition 4-15. Let us consider the set of subrings in $\mathfrak{D}^t$

$$\mathfrak{B} := \{L \in \mathcal{L}_2^t \mid \exists W \in \mathfrak{W}^t, \exists \mathcal{A}^t \in \mathfrak{D}^t \text{ and } \ \exists \mathcal{A}^c \in \overline{\mathfrak{A}^c} \text{ such that } \mathcal{A}^t = W\mathcal{A}^c W^{-1} \text{ and } L = W\partial_1^2 W^{-1}\ \}.$$

Since solving the KdV equations are an initial value problem, for an arbitrary initial state $u \in \mathbb{C}[[t_1]]$ we can find the time-development obeying the KdV equations. Thus we have $\mathfrak{B} \subset \mathbb{M}_{\mathrm{KdV}}$. On the other hand, from the definition, we can find $\mathbb{C}[\partial_1^2] \in \overline{\mathfrak{A}^c}$ which gives $\mathbb{N}_{\mathbb{C}[\partial_1^2]} = 2\mathbb{N}$. Further for an arbitrary $L \in \mathbb{M}_{\mathrm{KdV}}$, there is a gauge transformation, $W \in \mathfrak{W}^t$ such that $W^{-1}LW = \partial_1^2$ due to Lemma 4-8 (2). Hence $\mathfrak{B} \supset \mathbb{M}_{\mathrm{KdV}}$ and then $\mathfrak{B} \equiv \mathbb{M}_{\mathrm{KdV}}$. Such a consideration is justified by the direct limit and graded topology of $\mathfrak{D}^t$ or $\mathfrak{E}^c$.

Thus $\mathbb{M}_{\mathrm{KdV}}$ has naturally the topology induced from the linear topology of the micro-differential operator in Proposition 4-7 and the filter of $\mathbb{C}[\partial_s^2]$-module in Proposition 4-15, even though $\mathbb{M}_{\mathrm{KdV}}$ itself is not vector space.

4-27. Proposition.

(1) $\mathbb{M}_{\mathrm{KdV}}$ *is a filter space.*

(2) *The set of finite g type solutions of the KdV equation is denoted by* $\mathbb{M}_{\mathrm{KdV}g} := F_g\mathbb{M}_{\mathrm{KdV}} \setminus F_{g-1}\mathbb{M}_{\mathrm{KdV}}$ *and the set of finite type solutions of the KdV equation is denoted by* $\mathbb{M}_{\mathrm{KdV}\mathrm{finite}}$. *Then we have decomposition,*

$$\mathbb{M}_{\mathrm{KdV}\mathrm{finite}} = \coprod_{g<\infty} \mathbb{M}_{\mathrm{KdV}g}.$$

(3) *In the sense of Proposition 4-15 (3), it converges.*

$$\mathbb{M}_{\mathrm{KdV}} = \overline{\cup_{g=1}^{\infty} F_g\mathbb{M}_{\mathrm{KdV}}}.$$

For a point of $\mathfrak{M}_{\mathrm{KdV}g}$, they have non-trivial (effective) differential equations,

$$[\partial_n - 2^{2(n-1)} L^{(2n-1)/2}{}_+, L] = 0, \quad (n = 1, \cdots, g).$$

For the orbital as a dynamical system, we find a natural volume form $< dt_1, dt_2, \cdots, dt_g >_{\mathbb{C}}$.

4-28. Lemma.

(1) *For* $L_1, L_2 \in \mathbb{M}_{\mathrm{KdV}g}$*, there is* $W \in \mathfrak{W}^t$ *such that*

$$WL_1W^{-1} = L_2.$$

(2) *For a subset of* $\mathfrak{W}^f$ $(g > 0)$,

$$\mathfrak{W}^f{}_g := \{W \in \mathfrak{W}^f \mid WLW^{-1} \in \mathbb{M}_{\mathrm{KdV}g},\ \mathit{for}\ L \in \mathbb{M}_{\mathrm{KdV}g}\ \},$$

the projection π^g_{KdV} *along the orbit space in the quotient space of* $\mathbb{M}_{\mathrm{KdV}g}$ *by the action of* $\mathfrak{W}^f{}_g$ *is given by,*

$$\pi^g_{KdV}(\mathbb{M}_{\mathrm{KdV}g}/\mathfrak{W}^f{}_g) \sim \mathrm{pt}.$$

(3) *For*

$$\mathfrak{W}^f{}_{0,1} := \{W \in \mathfrak{W}^f \mid WLW^{-1} \in \mathbb{M}_{\mathrm{KdV}0} \cup \mathbb{M}_{\mathrm{KdV}1},\ \mathit{for}\ L \in \mathbb{M}_{\mathrm{KdV}0} \cup \mathbb{M}_{\mathrm{KdV}1}\ \},$$

the following relation holds,

$$\pi_{KdV}(\mathbb{M}_{\mathrm{KdV}0} \cup \mathbb{M}_{\mathrm{KdV}1}/\mathfrak{W}^f{}_{0,1}) \sim \mathrm{pt}.$$

Proof. (1) is essentially the same as Proposition 4-16 (2). The action of $\mathfrak{W}^f{}_g$ to $\mathbb{M}_{\mathrm{KdV}g}$ is transitive due to (1) and thus (2) is obtained.∎

We note that $\mathbb{M}_{\mathrm{KdV}0}$ should be regarded as a compactification of the base field $\mathbb{C}$, *i.e.*, $\mathbb{M}_{\mathrm{KdV}0} \equiv \mathbb{P}$ itself.

Now let us come back to the elastica problem. Firstly we note that the elastica problem is defined over the real functions. Hence we should restrict the above result to a real analytic problem. In other words, we choose a natural complex structure J ($J^2 = -1$) in the orbits space $< dt_1, dt_2, \cdots, dt_g >_{\mathbb{C}}$ and constraint it by $< dt_1, dt_2, \cdots, dt_g >_{\mathbb{R}}$ using the fact that finite g-type flow is a finite g-type solution of the KdV equation. Further the orbit satisfies the reality condition $|\partial_1\gamma| = 1$, which characterizes a certain type of hyperelliptic curves.

Secondly we should notice the difference of the categories of the previous chapter and this chapter. However as the g-type flow u is expressed by meromorphic functions over a hyperelliptic curve of genus g, elements of the *finite real* flow exist in the category of the formal power series. Hence the investigation of the finite flow does not depend on the difference.

Further the arc-length s corresponds to t_1 in the above argument but we are consider only the closed one. Hence firstly t_1 must be an element of $S^1 = \mathbb{R}/\mathbb{Z}$. Even though $u(s) \equiv \{\gamma, s\}_{\mathrm{SD}}$ is periodic, γ is not in general. We should restrict the space of the solution space of the KdV equation so that $\gamma(0) = \gamma(2\pi)$ or $\gamma(0) = \gamma(\infty)$.

We will define a projective structure in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g}$ by $\pi^g_{\mathrm{elas}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g} \to \mathfrak{M}^{\mathbb{P}}_{\mathrm{elas}\,g}$ so that for a point p in $\mathfrak{M}^{\mathbb{P}}_{\mathrm{elas}\,g}$, ${\pi^g_{\mathrm{elas}}}^{-1}(p)$ is the real number orbit, and let $\mathfrak{M}^{\mathbb{P}}_{\mathrm{elas\,finite}} = \coprod_g \mathfrak{M}^{\mathbb{P}}_{\mathrm{elas}\,g}$ as did in Definition 3-18.

We summary these results in the following proposition.

4-29. Proposition. *There are natural injections*

$$i_{\mathrm{KdV}} : \mathbb{M}^{\mathbb{P}}_{\mathrm{elas\,finite}} \hookrightarrow \mathbb{M}_{\mathrm{KdV\,finite}}, \quad \iota_{\mathrm{KdV}} : \mathfrak{M}^{\mathbb{P}}_{\mathrm{elas\,finite}} \hookrightarrow \mathfrak{M}_{\mathrm{KdV\,finite}}$$

which satisfy

(1)

$$\iota_{\mathrm{KdV}} \circ \pi_{\mathrm{elas}} = \pi_{\mathrm{KdV}} \circ i_{\mathrm{KdV}},$$

(2)

$$\mathbb{M}_{\mathrm{KdV}} \setminus \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \neq \emptyset.$$

Using the above results, there is a filtration in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ such that

$$F_g\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} := \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \cap F_g\mathbb{M}_{\mathrm{KdV}},$$

which satisfies

$$F_g\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \subset F_{g+1}\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}, \quad \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g} = F_g\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}/F_{g-1}\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}.$$

We have written just $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ as $i_{\mathrm{KdV}}(\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}})$ and $\mathfrak{M}^{\mathbb{P}}_{\mathrm{elas}}$ as $\iota_{\mathrm{KdV}}(\mathfrak{M}^{\mathbb{P}}_{\mathrm{elas}})$ for brevity.

Next we will consider the real orbits or the "time" development of each finite g type flow in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ (instead of $\mathbb{M}_{\mathrm{KdV}}$). Let us recall the fact that rational points in $[0, 1)$, *i.e.*, $\mathbb{Q}/\mathbb{Z}$, are measure zero in $[0, 1)$. Further it is known that for a torus $\mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$, a real direct line (orbit) stemmed from the origin with an angle θ does not stand upon the origin again if $\theta \notin \tan^{-1}(\mathbb{Q}/\mathbb{Z})$. Similarly in general, the real number "time" development of the finite g type solution is not periodic in "time" t_i $(i > 1)$, in the g-dimensional torus $\mathcal{J}_g$ which is called quasi-periodic solutions. Hence we conclude that such an orbit is homeomorphic to $\mathbb{R}^{g-1}$ in this sense and show the following proposition.

4-30. Proposition. *For each* pt $\in \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g}$, *we have a restricted action of* $\mathfrak{W}^f{}_g$ *and thus the following results are satisfied:*

(1)

$$\pi^g_{\mathrm{KdV}}|_{\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g}}(\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g}/[\mathfrak{W}^f{}_g|_{\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g}}]) = \mathrm{pt}$$

(2)

$$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g}/[\mathfrak{W}^f{}_g|_{\mathfrak{M}^{\mathbb{P}}_{\mathrm{elas}\,g}}] \approx S^1 \times \mathbb{R}^{g-1}, \quad \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}\,g}/[\mathfrak{W}^f{}_g|_{\mathfrak{M}^{\mathbb{P}}_{\mathrm{elas}\,g}}] \approx \mathbb{R}^{g-1}, \qquad \textit{for } g > 1.$$

(3)

$$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}0} \cup \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}1} / \mathfrak{W}^f{}_{0,1}|_{\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}0}\cup \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}1}} \approx S^1, \quad (\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}0} \cup \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}1} / \mathfrak{W}^f{}_{0,1}|_{\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}0}\cup \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}1}})/\mathrm{Isom}(S^1) \approx \mathrm{pt}.$$

We will recover the base ring with smooth functions. In other words, we show that the completion in Proposition 4-27 can be extended to $\mathfrak{E}^s$ because the convergence is determined only by the topology of order of the differential operator as shown in the following lemma.

4-31. Lemma.

(1) *When we define* $\mathfrak{E}^s{}_m := \{D \in \mathfrak{E}^s \mid \mathrm{deg} D \le m\ \}$, $\mathfrak{E}^s$ *has a filter topology,*

$$\mathfrak{E}^s = \cup_m \mathfrak{E}^s{}_m, \quad \{0\} = \cap_m \mathfrak{E}^s{}_m, \quad \mathfrak{E}^s{}_m \subset \mathfrak{E}^s{}_{m+1}.$$

(2) $\mathfrak{E}^s$ *is a linear topological space with respect to this filter topology.*

Due to Lemma 4-31 and note below the Proposition 4-3, we have the following proposition.

4-32. Proposition. *Let us define the moduli space of the KdV equations over the ring of the smooth functions:*

$$\mathbb{M}_{\mathrm{KdV}}{}^{\infty} := \{u \in \mathcal{C}^\infty(\mathcal{V}^\infty) \mid \partial_{t_n} u - \Omega_1^{n-1} u = 0 \ \mathit{for}\ \ \forall n\ \}, \quad \mathcal{M}_{\mathrm{KdV}}{}^{\infty} := \mathbb{M}_{\mathrm{KdV}}{}^{\infty}/(t_1).$$

Then

(1) $\mathbb{M}_{\mathrm{KdV}}$ *is dense in* $\mathbb{M}_{\mathrm{KdV}}{}^{\infty}$.
(2) $\mathbb{M}_{\mathrm{KdV finite}}$ *is a subset of* $\mathbb{M}_{\mathrm{KdV}}{}^{\infty}$.

Proof. Due to the Weierstrass preparation theorem, for an arbitrary germ in $\underline{\mathcal{C}^\infty}(\mathbb{R})$, there is a sequence in $\underline{\mathcal{F}}(\mathbb{R})$ conversing it. Integrability due to Proposition 3-15 asserts that the difference does not enlarge for the time development. Hence (1) is proved. (2) is obvious ∎

Hence we have the finial statement in this section.

4-33. Proposition. $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} \subset \mathbb{M}_{\mathrm{KdV}}{}^{\infty}$ *has the filter topology induced from* $\mathbb{M}_{\mathrm{KdV}}{}^{\infty}$.

(1) *There is a natural decomposition,*

$$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas,finite}} = \coprod_{g<\infty} \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g}.$$

(2) *With respect to the induced topology and in the sense of Propositions 4-27 (3) and 4-32 (1), we have*

$$\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}} = \overline{\cup_{g=1}^\infty F_g \mathbb{M}^{\mathbb{P}}_{\mathrm{elas}\,g}}.$$

§5. Algebro-Geometric Properties of the KdV flow II –Analytic Expressions–

In this section, we will give a more concrete argument. For example, we will give another proof of Theorem 4-2 and Proposition 4-33 at Proposition 5-22. This section is based on the inverse scattering method, Krichever's scheme and Baker's approach.

The studies of the KdV equation have a long history. There were so many researchers contributing them, *e.g.*, Miura, Gardner, Greene, Kruskal, Lax, and so on [D, DJ]. Owing to their studies, we will give, here, another aspect of the KdV equation without proofs, which is called the inverse scattering method.

5-1. Proposition. [BBEIM, D, DJ, Kr]

(1) *For a solution u of the n-th KdV equation, there is a complex valued smooth function $\psi_{\overline{x}}$ over $\mathbb{R} \times \{t_n\}$, which is a universal covering of S^1 ($S^1 = \mathbb{R}/2\pi\mathbb{Z}$) and $\{t_n\} \subset \mathbb{R}$,*

$$\psi_{\overline{x}} \in \mathcal{C}^\infty(\mathbb{R} \times \{t_n\}, \mathbb{C}),$$

as an eigen vector of the eigenvalue problem over $\mathbb{R}$ ($S^1 = \mathbb{R}/2\pi\mathbb{Z}$),

$$L = \partial_1^2 + u, \quad -L\psi_{\overline{x}} = \overline{x}\psi_{\overline{x}},$$

and as a solution of

$$(\partial_{t_n} - 2^{2(n-1)} L^{(2n-1)/2}{}_+)\psi_{\overline{x}} = 0.$$

The deformation of u with respect to t_n which preserves the eigen value $\overline{x}$ is equivalent with that u is a solution of the n-th KdV-equation and vice-versa. Here the extension of the domain of u over S^1 to $\mathbb{R}$ is naturally defined as $u(t_1) = u(t_1 + 2\pi)$. These equations are equivalent to the Lax equations in Definition 4-20.

5-2. Remark.

(1) The eigenvalue problem $-L\psi_{\overline{x}} = \overline{x}\psi_{\overline{x}}$ can be regarded as a quantization of the "classical" equation

$$(-\partial_1^2 - \frac{1}{2}\{\gamma, s\}_{\mathrm{SD}}(t_1))\psi(t_1) = 0.$$

Indeed, $(\partial_\tau - L)\Psi = 0$, ($\Psi = \exp(\tau\overline{x})\psi$) appears when we quantize $\psi(t_1)$ by means of the path integration [R,Mat0].

(2) For finite type solutions of the KdV hierarchy, the Lax equations and the compatibility condition are essentially reduced to finite relations. Due to Lemma 4-1 and Definition 4-24, the equations with respect to t_m $m > N$ are trivial one for N-type solution.

For a while, we will assume that u is real. Let $\mathrm{Spect}(-L)$ denote a set of $\overline{x}$. Due to hermitian properties of $-L$, $\mathrm{Spect}(-L)$ is a subset of real number bounded from below. The function $\psi_{\overline{x}}(t_1)$ is regarded as a section of line bundle over $\mathrm{Spect}(-L)$.

For bases y_0 and y_1 of the solution space of $-L\psi_{\overline{x}} = \overline{x}\psi_{\overline{x}}$, ($\psi_{\overline{x}} = ay_0 + by_1$, for a, $b \in \mathbb{C}$),

$$y_0(0, \overline{x}) = 1, \quad y_1(0, \overline{x}) = 0, \quad \partial_1 y_0(0, \overline{x}) = 0, \quad \partial_1 y_1(0, \overline{x}) = 1,$$

we have monodoromy matrix defined as

$$M(\overline{x}) := \begin{pmatrix} y_0(\pi, \overline{x}) & y_1(\pi, \overline{x}) \\ \partial_1 y_0(\pi, \overline{x}) & \partial_1 y_1(\pi, \overline{x}) \end{pmatrix},$$

whose determinant is unity. If the eigenvalue of this matrix ρ is in the unit circle in $\mathbb{C}$ ($|\rho| = 1$), the solution $\psi_{\overline{x}}$ is called stable and exist as a global section over the line bundle over $s \in \mathbb{R}$. Unless, it is called unstable and it means that there is no global section over $s \in \mathbb{R}$ even though we can find local solutions of $-L\psi_{\overline{x}} = \overline{x}\psi_{\overline{x}}$. We sometimes refer the unstable state "gap state" or "forbidden state". The determinant whether it is stable or unstable is done by the characteristic equation,

$$\rho^2 - \Delta_u \rho + 1 = 0,$$

where $\Delta_u := \mathrm{tr} M$. If its discriminant $\Delta_u^2 - 4$ is non-positive, corresponding $\overline{x}$ becomes stable.

Since $\Delta_u^2 - 4$ is an analytic function over $\mathrm{Spect}(-L) - \{\infty\}$ and has ordered zero points $\overline{x}_1, \overline{x}_2 \cdots$, it has infinite product expression:

$$(\Delta_u^2 - 4) = c \prod_{j=0}^{\infty} (\overline{x} - \overline{x}_j),$$

where c is a constant in $\overline{x}$. This fact is correct even for the case that u is complex valued and thus we will return to the general u form here.

5-3. Proposition. [MM] *For $-L\psi_{\overline{x}} = \overline{x}\psi_{\overline{x}}$ with smooth $u(t_1)$ over $\mathbb{R}$, the discriminant Δ is characterized by infinite $\overline{x}_j$ and can be rewritten as,*

$$(\Delta_u^2 - 4) = \left(\prod_{j=0,\text{single zeros}} (\overline{x} - \overline{x}_j) \right) h(\overline{x})^2,$$

where $h(\overline{x}) = \sqrt{c} \prod_{j',\text{double zeros}}^{\infty} (\overline{x} - \overline{x}_j)$ is the part of double zeros.

For large $\overline{x}$, $-L$ asymptotically behaves like $-\partial_1^2$ for bounded u and thus the asymptotic behavior of Δ can be investigated. Since the ground state corresponds to a single zero of $\Delta^2 - 4$ and other each gap has two single zeros of $\Delta^2 - 4$, the number of single zeros of $\Delta^2 - 4$ must be odd. Here we will consider a case with finite single zero points $2g + 1$:

$$\frac{(\Delta_u^2 - 4)}{h(\overline{x})^2} = \prod_{j=1}^{2g+1} (\overline{x} - \overline{x}_j).$$

We refer such a case as finite-gap-state. It should be noted that $\psi_{\overline{x}}$ has natural involution π : $\mathrm{Spect}(-L) \longrightarrow \mathrm{Spect}(-L)$ ($\pi : \overline{y} \longrightarrow -\overline{y}$, $\pi : \infty = \infty$) where $\overline{y} = \sqrt{\Delta_u^2 - 4}/h(\overline{x})$. Due to analyticity, we can extend $\mathrm{Spect}(-L)$ to complex. As for $u \equiv 0$ case, $\mathrm{Spect}(-\partial_1^2)$ is complexfied to $\mathbb{P}$ (even though we need more precise arguments), the energy spectrum $\mathrm{Spect}(-L)$ is, in general, reduced to a hyperelliptic curve C_g due to its two-folding property. In fact for $\overline{y} = \sqrt{\Delta_u^2 - 4}/h(\overline{x})$, this relation means a hyperelliptic curve defined in 4-13.

In this section, we will fix a hyperelliptic curve C_g with genus g given by an affine curve,

$$\overline{y}^2 = h_g(\overline{x}, 1) = (\overline{x} - c_1) \cdots (\overline{x} - c_{2g+1}).$$

In other words, we deal with a commutative ring $\mathbb{C}[\overline{x}, \overline{y}]/(\overline{y}^2 - h_g(\overline{x}, 1)) \cup \{\infty\}$. We should note that for a hyperelliptic curve C_g, there exists a differential operator $-L$ with u such that its spectrum $\mathrm{Spect}(-L)$ gives the hyperelliptic curve isomorphic to C_g.

5-4. Proposition. *Let the moduli space of hyperelliptic curves of genus g be denoted by $\mathfrak{M}_{\mathrm{hyp},g}$. Then $\mathfrak{M}_{\mathrm{hyp},g}$ is $(2g-1)$ dimensional space.*

Proof. A point in the moduli space $\mathfrak{M}_{\mathrm{hyp},g}$ is characterized by $2g+1$ zero points of $h_g(x,1)$ in the above definition and ∞ point. However in these variables, there are several symmetries which express the same compact Riemannian surface. First one is translational symmetry $c_j \to c_j + \alpha_0$, $\alpha_0 \in \mathbb{C}$. Second one is dilatation $c_j \to c_j\alpha_1$ $\alpha_1 \in \mathbb{C}$. Third one is $(\overline{x}, \overline{y}) \to (1/\overline{x}, \overline{y}\prod_j c_j/\overline{x}^{(2g+1)/2})$, which reduces $c_j \to 1/c_j$. Hence the remainder degree of freedom is $2g-1$.

5-5. Remark. We will mention $\mathfrak{M}_{\mathrm{hyp},g}$ here. We consider a smooth curve in $\mathfrak{M}_{\mathrm{hyp},g}$ which is not degenerated; $c_i \neq c_j$ if $i \neq j$ and all of c_j are finite value of $\mathbb{C}$. Let us find the largest distance $|c_j - c_k|$ of pair (c_j, c_k) in $\{c_j\}$ as an arbitrary $|c_j - c_k|$ does not vanish because the curve is not degenerated. Let us rename them as (c_1, c_{2g+1}) and define

$$(\alpha_1, \cdots, \alpha_{2g-1}) := ((c_2 - c_1)/(c_{2g+1} - c_1), \cdots, (c_{2g} - c_1)/(c_{2g+1} - c_1)) \in \mathbb{C}^{2g-1}.$$

Since $1 - \alpha_j = (c_{j+1} - c_{2g+1})/(c_1 - c_{2g+1})$ and $|c_1 - c_{2g+1}|$ is the largest distance, the region of each α_j must be constrained as $|\alpha_j| \leq 1$ and $|1 - \alpha_j| \leq 1$. Next we will order α following the law,

(1) if $Re(\alpha_i) < Re(\alpha_j)$, $i < j$.
(2) if $Re(\alpha_i) = Re(\alpha_j)$ and $Im(\alpha_i) < Im(\alpha_j)$, $i < j$.

Hyperelliptic curves of genus g are determined as two-fold coverings of $\mathbb{P}^1$ ramified at 0, 1, ∞ and $2g-1$ additional points as the above order.

However it is difficult to deal with deformation from non-degenerate hyperelliptic curves to degenerate curves [HM, IUN, Mum0].

5-6. Definition. *[BBEIM, IUN, Kr, Mum0-2]*

(1) Let

$$\mathrm{H}_1(C_g, \mathbb{Z}) = \bigoplus_{j=1}^{g} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z}\beta_j.$$

denote the homology of an algebraic curve C_g.

(2) We introduce the periodic matrix of the curve C_g, in terms of the normalized first kind one-form ω_i over C_g:

$$1 = \left[\int_{\alpha_j} \hat{\omega}_i\right], \quad \mathbb{T} = \left[\int_{\beta_j} \hat{\omega}_i\right], \quad \hat{\boldsymbol{\Omega}}_{\mathbf{1}} = \begin{bmatrix} 1 \\ \mathbb{T} \end{bmatrix}.$$

(3) For fixing $\mathbb{T}$, we define the theta function $\theta : \mathbb{C}^g \longrightarrow \mathbb{C}$ by,

$$\theta(z) := \theta(z|\mathbb{T}) := \sum_{n \in \mathbb{Z}^g} \exp\left[2\pi i \left\{\frac{1}{2}\ {}^t n \mathbb{T} n + \ {}^t n z\right\}\right].$$

5-7. Proportions. [BBEIM, IUM, Kr, Mum0-2]

(1) *By defining the Abel map for g-th symmetric product of the curve C_g,*

$$\hat{w} : \mathrm{Sym}^g(C_g) \longrightarrow \mathbb{C}^g, \quad \left(\hat{w}_k(Q) := \sum_{i=1}^{g} \int_{\infty}^{Q_i} s\hat{\omega}_k\right),$$

the Jacobi variety $\hat{\mathcal{J}}_g$ is realized as a complex torus,

$$\hat{\mathcal{J}}_g = \mathbb{C}^g/\hat{\mathbf{\Lambda}}.$$

Here $\hat{\mathbf{\Lambda}}$ is a lattice generated by $\hat{\mathbf{\Omega}}$. For the Abelian group of the divisor of the line bundle over a hyperelliptic curve C_g, which is called Picard group $\mathrm{Pic}^0(X)$, the Abel theorem is expressed by $\mathrm{Pic}^0(X) \approx \mathbb{C}^g/\hat{\mathbf{\Lambda}}$.

(2) *The theta function has monodoromy properties*

$$\theta(z + e_k) = \theta(z), \quad \theta(z + \tau_k) = \mathrm{e}^{-2\pi i z_k + \pi i \tau_{kk}}\theta(z).$$

(3) *The Riemann theorem gives that*

$$\theta(\hat{w}(Q) - \sum_{i=1}^{g} \hat{w}(P_g) + K) \not\equiv 0,$$

where K is a constant called Riemann constant if and only if P_g's are general points on C_g.

As $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas},g}$ and $\mathbb{M}_{\mathrm{KdV},g}$ have the natural projections, we will introduce the universal family of hyperelliptic curves of $\mathfrak{M}_{\mathrm{hyp},g}$ induced from $\pi_{\mathrm{hyp}} : \mathcal{J}_g \mapsto \mathrm{pt} \in \mathfrak{M}_{\mathrm{hyp},g}$.

5-8. Proposition. (Krichever, Mulase)[Kr, Mul, Mum1]

(1) *A finite g type solution of the KdV equation is given by a meromorphic function over the Jacobi variety $\mathcal{J}_g$ of a hyperelliptic curves C_g.*

(2) *There is a natural bijection between the moduli spaces of hyperelliptic curves $\mathfrak{M}_{\mathrm{hyp},g}$ and $\mathfrak{M}_{\mathrm{KdV},g}$,*

$$\mathfrak{M}_{\mathrm{hyp},g} \approx \mathfrak{M}_{\mathrm{KdV},g}.$$

As (2) comes from the previous section, we will mention its idea of (1) as follows [Kr, SW]. Krichever started with ψ_x, a solution of $(-\partial_1^2 - u + x^2)\psi_x = 0$, which is called the Baker-Akhiezer function. His approach is very natural in the soliton theory and can be generalized from the case of the KdV hierarchy, which is related to hyperelliptic curves, to that in the KP hierarchy related to more general compact Riemannian surfaces.

5-9. Lemma. [Kr]

(1) *For a solution of the KdV equation whose $\mathrm{Spect}(-L)$ is associated with the hyperelliptic curve C_g, we parameterize the eigenvalue $-x^2$ for $L\psi_x = x^2\psi_x$. Then $1/x$ is a local parameter of ∞ of C_g.*

(2) *ψ_x is meromorphic on $C_g - \infty$ and at the point ∞ it has an essential singularity*

$$\psi_x = \mathrm{e}^{sx}\psi_W, \quad \psi_W := (1 + \sum_{i=1}^{\infty} a_i(t_1)x^{-i}).$$

Here this expansion gives us the recursive relation $-2\partial_1 a_i = -La_{i-1}$with $a_0 = 1$.

Proof. (1): For a sufficiently large $|x|$, this equation can be approximated by $(-\partial_1^2 + x^2)\psi_x \sim 0$. Thus we can regard $\psi_x \sim \exp(sx)$. In other words for a local coordinate $z = 1/x$ around $\infty \in \mathrm{Spect}(-L)$, $\psi_x \sim \exp(-s/z)(1 + \mathcal{O}(z))$: $1/x^2 = 1/\overline{x}$ is a local coordinate around $\infty \in \mathrm{Spect}(-L)$. (2) can be obtained by straightforward computations.

Using this Lemma 5-9, we follow the Krichever's construction of the finite g type solution. As we gave the Jacobi varieties and theta functions of hyperelliptic curve C_g in 4-13 and Proposition 5-7, we introduce a normalized Abelian differential of the second kind, $\hat{\eta}_{P,i}$,

$$\hat{\eta}_{P,n} = d\left(\frac{1}{t^{n-1}} + \mathcal{O}(1)\right),$$

around P using a local parameter t $(t(P) = 0)$ with the normalization

$$\int_{\alpha_j} \hat{\eta}_{P,n} = 0, \qquad \text{for } j = 1, \cdots, g.$$

As we have prepare to express the Baker-Akhiezer function, we consider the deformation equation,

$$(\partial_{t_n} - 2^{2(n-1)} L_+^{(2n-1)/2})\psi_x = 0.$$

Since $z = 1/x$ is a local parameter around ∞ and around there $L_+^{(2n-1)/2} \sim \partial_1^{(2n-1)}$, we introduce

$$\hat{\eta}_{\infty,n} = d(x^{2n-1} + \mathcal{O}(1)),$$

and consider the function

$$\mathcal{E}(t, Q) = \exp\left(\sum_{\alpha,j} 2^{2(n-1)} t_{\alpha,j} \int^Q \eta_{P_\alpha,i}\right).$$

Around ∞, $\mathcal{E}(t,Q) \sim \exp(\sum_{n=1}^\infty 2^{2(n-1)} t_n x^{2n-1})$ and $\partial_{t_n}\mathcal{E}(t,Q) \sim 2^{2(n-1)} x^{2n-1}\mathcal{E}(t,Q)$. Due to Lemma 4-8 and 5-9 and by letting $L = W(s,\partial_1)\partial_1^2 W(s,\partial_1)^{-1}$ in the sense of Lemma 4-14, we obtain the relations $\psi_x = W(s,\partial_1)\mathcal{E}(t,Q) + \mathcal{O}\left(\frac{1}{x}\right)$ and

$$L^{n/2}\, W(s,\partial_1)\, \mathcal{E}(t,Q) = W(s,\partial_1)\, \partial_1^n \mathcal{E}(t,Q) + \mathcal{O}\left(\frac{1}{x}\right).$$

From the Lax equations in Proposition 5-1, ψ_x is expressed by $\psi_x/\mathcal{E} = (\psi_x/\mathcal{E})(xt_1, 4x^3 t_2, 8x^5 t_3, \cdots) + \mathcal{O}(\frac{1}{x})$.

On the other hand, even though $\mathcal{E}(t,Q)$ is satisfied with the dispersion relation around ∞ and has no monodoromy around α_j's, it has monodoromy around β_j

$$\exp(2\pi i U_j) := \exp(\sum_{j,\alpha} 2^{2(j-1)} t_j H^i{}_{\alpha,j})$$

where

$$H^i{}_{\alpha,j} = \frac{1}{2\pi i}\int_{\beta_i} d\hat{\eta}_{P_\alpha,j}.$$

Noting this monodoromy of the theta function in Proposition 5-7, we can find a single value function over $\mathbb{C}^g$, which is known as Baker-Akhiezer function;

$$\psi_x = \mathcal{E}(t,Q)\frac{\theta(w(Q) + \sum_{\alpha,j} 2^{2(j-1)} t_{\alpha,j} H_{\alpha,j} - \sum_{i=1}^g w(P_g) + K)}{\theta(w(Q) - \sum_{i=1}^g w(P_g) + K)}.$$

This is a solution of the Lax equations in Proposition 5-1. We can find a finite type solution of the KdV equation by using the zero mode using Proposition 2-8. ψ_x is determined by an analysis on the functions over $\mathbb{C}^g$ related to the Jacobi variety $\hat{\mathcal{J}}_g$. As the map form C_g's to the Jacobi variety $\hat{\mathcal{J}}_g$ is known as Abel map, finding inverse map from functions over $\hat{\mathcal{J}}_g$ to functions over C_g's is known as Jacobi inverse problem. Krichever's scheme should be regarded as the Jacobi inverse method and can be applied even to a generalized Jacobi variety. It shows the existence of an injection form $\mathfrak{M}_{\mathrm{hyp}}$ to $\mathfrak{M}_{\mathrm{KdV}}$,

5-10. Remark. For a finite type solution of the KdV hierarchy u, we have the hyperelliptic curve C_g as a spectrum of $-L$ to u. Then the above arguments give the following results:

(1) The orbit of the equations of the KdV hierarchy is realized in a direct line in the Jacobi variety J_g of C_g.
(2) Any finite g solution u is given as a solution of the Jacobi inverse problem of the Jacobi variety J_g.

As far as we will deal with only hyperelliptic curves and the KdV hierarchy, we can give more concrete arguments based upon Baker's original argument [Ba1, Ba2].

5-11. Definition. [Ba1, Ba2, BEL] We introduce the family of the differential forms:

(1)

$$\omega_1 = \frac{d\overline{x}}{2\overline{y}}, \quad \omega_2 = \frac{\overline{x}d\overline{x}}{2\overline{y}}, \quad \cdots \quad \omega_g = \frac{\overline{x}^{g-1}d\overline{x}}{2\overline{y}}.$$

(2)

$$\eta_j = \frac{1}{2\overline{y}}\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+1+j}\overline{x}^k d\overline{x}, \quad (j = 1, \cdots, g).$$

5-12. Lemma. [Ba1, Ba2, BEL]

(1) *ω's are the basis of the holomorphic function valued cohomology of hyperelliptic curve C_g, which give unnormalized periods:*

$$\mathbf{\Omega}' = \left[\int_{\alpha_j} \omega_i \right], \quad \mathbf{\Omega}'' = \left[\int_{\beta_j} \omega_i \right], \quad \mathbf{\Omega} = \begin{bmatrix} \mathbf{\Omega}' \\ \mathbf{\Omega}'' \end{bmatrix}.$$

(2) *They are related to the normalized ones:*

$${}^t[\widehat{\omega}_1 \cdots \quad \widehat{\omega}_g] := \mathbf{\Omega}'^{-1}\ {}^t[\omega_1 \quad \cdots \omega_g], \quad \mathbb{T} := \mathbf{\Omega}'^{-1}\mathbf{\Omega}''.$$

(3) *η's are the unnormalized one-form of the second kind over C_g and then the complete hyperelliptic integral of the second kinds is given as*

$$H' := \left[\int_{\alpha_j} \eta_i \right], \quad H'' := \left[\int_{\beta_j} \eta_i \right].$$

Here the contours in the integral are, for example, given in p.3.83 in [Mum2].

Proof. We check holomophicity of the forms in (1) and (3). A zero point of $\overline{y} = 0$, or a root c_j of $f(\overline{x}) = 0$, corresponds to a point $(c_j, 0)$ of the curve C_g. We use a local coordinate $z^2 := (\overline{x} - c_j)$ and $\overline{x}^m d\overline{x}/(2\overline{y}) \sim (z^2 + c_j)^m dz + \cdots$. On the other hand, around ∞ point, let us choose local coordinate $1/x$ as $1/x^2 = 1/\overline{x}$ and then $\overline{x}^m d\overline{x}/(2\overline{y}) \sim (1/x)^{2g-2m+2} dx + \cdots$. Hence ω is holomorphic all over the curve C_g while η is holomorphic except ∞ point.∎

5-13. Definition.

(1) The unnormalized Jacobi variety $\mathcal{J}_g$ is defined by a complex torus,

$$\mathcal{J}_g = \mathbb{C}^g / \mathbf{\Lambda},$$

where $\mathbf{\Lambda}$ is a lattice generated by $\mathbf{\Omega}$.

(2) We defined the theta function by,

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z) = \theta \begin{bmatrix} a \\ b \end{bmatrix} (z; \mathbb{T}) = \sum_{n \in \mathbb{Z}^g} \exp \left[2\pi i \left\{ \frac{1}{2}\ {}^t(n+a)\mathbb{T}(n+a) + {}^t(n+a)(z+b) \right\} \right].$$

5-14. Proposition. *The Riemann constant of the hyperelliptic curve C_g is given as*

$$K = \sum_{j=1}^{g} \int_{\infty}^{\mathrm{A}_j} \widehat{\omega} = \delta' + \delta''\mathbb{T}$$

where $\delta' = {}^t\left[\frac{g}{2} \quad \frac{g-1}{2} \quad \cdots \quad \frac{1}{2} \right], \quad \delta'' = {}^t\left[\frac{1}{2} \quad \cdots \quad \frac{1}{2} \right].$

Proof. This proof is in p.3.82 in [Mum2].

Using the Abel map $\mathrm{Sym}^g(C_g) \longrightarrow \mathbb{C}^g$, we define the coordinate in $\mathbb{C}^g$,

$$\mathfrak{t}_j := \sum_{i=1}^{g} \int^{(\overline{y}_i, \overline{x}_i)} \omega_j.$$

Here we note that $\mathfrak{t}_j$ behaves $(1/x)^{2(g-j)+1}$ around ∞ point if we use the parameter $x^2 = \overline{x}$.

5-15. Definition. ($\wp$-function, Baker)[Ba1,Ba2]

(1) Using the coordinate $\mathfrak{t}_j$, the σ-function, which is a holomorphic function over $\mathbb{C}^g$, is defined by

$$\sigma(\mathfrak{t}) := \sigma(\mathfrak{t}; C_g) := \exp(-\frac{1}{2}\ {}^t\ \mathfrak{t}H'{\boldsymbol{\Omega}'}^{-1}\mathfrak{t})\vartheta\begin{bmatrix}\delta'' \\ \delta'\end{bmatrix}({\boldsymbol{\Omega}'}^{-1}\mathfrak{t}; \mathbb{T}).$$

(2) In terms of the σ-function, the hyperelliptic $\wp$-function over the hyperelliptic curve C_g is defined by

$$\wp_{ij}(\mathfrak{t}) := -\frac{\partial^2}{\partial \mathfrak{t}_i \partial \mathfrak{t}_j} \log \sigma(u) = \frac{\sigma_i(\mathfrak{t})\sigma_j(\mathfrak{t}) - \sigma_{ij}(\mathfrak{t})\sigma(\mathfrak{t})}{\sigma(\mathfrak{t})^2}.$$

As σ-function is an entire function over $\mathbb{C}^g$ and has single zero at $g-1$ dimensional subvariety of $\mathbb{C}^g$ which is called theta-divisor, the hyperelliptic $\wp_{ij}$ has the second order singularity and function of $\mathcal{J}_g$.

5-16. Remark. It is worth while noting that from Definition 5-15, the hyperelliptic $\wp$-function can be concretely computed for a given hyperelliptic curve C_g. The summation in the definition of θ function rapidly converges due to effects of $\mathbb{T}$ and others are integrations of primary functions. Further it is known that $\wp_{gi}$ is an elementary symmetric function, *i.e.*, for $F(\overline{x}) = (\overline{x} - \overline{x}_1)(\overline{x} - \overline{x}_2)\cdots(\overline{x} - \overline{x}_g)$, [Ba1, BEL],

$$F(\overline{x}) = \overline{x}^g - \sum_{i=1}^{g} \wp_{gi}\overline{x}^{i-1}.$$

Accordingly, by numerical approach, we can compute a value of the hyperelliptic $\wp$ function as Euler determined a value of the elliptic integral to know the shape of a classical elastica by numerical method [E, L, T1, 2]. This approach was discovered by Baker about one hundred years ago [Ba1, 2, Mat7-10].

We emphasize that it completely differs from Krichever's approach based upon Baker-Akhiezer theorem explained in §4. Krichever's arguments might not give us practical algorithms to fix parameters of general hyperelliptic function except solutions expressed by elliptic or hyperbolic functions. (Due to its abstract, it is a good strategy to construct soliton theory.)

On the other hand, Baker's original method determines concrete function forms of corresponding $\wp$ functions, for any algebraically given hyperelliptic curves (even for degenerate curves in $\mathfrak{M}_{\mathrm{hyp},g}$). We can expand $\wp$-function around a general point and know its parameter dependence.

Since this Baker's construction in [Ba2] might be no longer in recent researchers' memory as long as I know, we believe that this review of Baker's work has meaning. We believe that it is very useful for the analysis in physics [BEL, Ma7-10].

In [Ba2] Baker found that the $\wp$-functions obey the following differential equations, which contain the KdV hierarchy.

5-17. Example. *(genus = 3)*[Ba2] *Let us express* $\wp_{ijk} := \partial\wp_{ij}(\mathrm{t})/\partial\mathrm{t}_k$ *and* $\wp_{ijkl} := \partial^2\wp_{ij}(\mathrm{t})/\partial\mathrm{t}_k\partial\mathrm{t}_l$. *The hyperelliptic* $\wp$*-function obeys the relations*

(1) $\wp_{3333} - 6\wp_{33}^2 = 2\lambda_5\lambda_7 + 4\lambda_6\wp_{33} + 4\lambda_7\wp_{32}$,

(2) $\wp_{3332} - 6\wp_{33}\wp_{32} = 4\lambda_6\wp_{32} + 2\lambda_7(3\wp_{31} - \wp_{22})$,

(3) $\wp_{3331} - 6\wp_{31}\wp_{33} = 4\lambda_6\wp_{31} - 2\lambda_7\wp_{21}$,

(4) $\wp_{3322} - 4\wp_{32}^2 - 2\wp_{33}\wp_{22} = 2\lambda_5\wp_{32} + 4\lambda_6\wp_{31} - 2\lambda_7\wp_{21}$,

(5) $\wp_{3321} - 2\wp_{33}\wp_{21} - 4\wp_{32}\wp_{31} = 2\lambda_5\wp_{31}$,

(6) $\wp_{3311} - 4\wp_{31}^2 - 2\wp_{33}\wp_{11} = 2\Delta_\wp$,

(7) $\wp_{3222} - 6\wp_{32}\wp_{22} = -4\lambda_2\lambda_7 - 2\lambda_3\wp_{33} + 4\lambda_4\wp_{32} + 4\lambda_5\wp_{31} - 6\lambda_7\wp_{11}$,

(8) $\wp_{3221} - 4\wp_{32}\wp_{21} - 2\wp_{31}\wp_{22} = -2\lambda_1\lambda_7 + 4\lambda_4\wp_{31} - 2\Delta_\wp$,

(9) $\wp_{3211} - 4\wp_{31}\wp_{21} - 2\wp_{32}\wp_{11} = -4\lambda_0\lambda_7 + 2\lambda_3\wp_{31}$,

(10) $\wp_{3111} - 6\wp_{31}\wp_{11} = 4\lambda_0\wp_{33} - 2\lambda_1\wp_{32} + 4\lambda_2\wp_{31}$,

(11) $\wp_{2222} - 6\wp_{22}^2$
$= -8\lambda_2\lambda_6 + 2\lambda_3\lambda_5 - 6\lambda_1\lambda_7 - 12\lambda_2\wp_{33} + 4\lambda_3\wp_{32} + 4\lambda_4\wp_{22} + 4\lambda_5\wp_{21} - 12\lambda_6\wp_{11} + 12\Delta_\wp$,

(12) $\wp_{2221} - 6\wp_{22}\wp_{21} = -4\lambda_1\lambda_6 - 8\lambda_0\lambda_7 - 6\lambda_1\wp_{33} + 4\lambda_3\wp_{31} + 4\lambda_4\wp_{21} - 2\lambda_5\wp_{11}$,

(13) $\wp_{2211} - 4\wp_{21}^2 - 2\wp_{22}\wp_{11} = -8\lambda_0\lambda_6 - 8\lambda_0\wp_{33} - 2\lambda_1\wp_{32} + 4\lambda_2\wp_{31} + 2\lambda_3\wp_{21}$,

(14) $\wp_{2111} - 6\wp_{21}\wp_{11} = -2\lambda_0\lambda_5 - 8\lambda_0\wp_{32} + 2\lambda_1(3\wp_{31} - \wp_{22}) + 4\lambda_2\wp_{21}$,

(15) $\wp_{1111} - 6\wp_{11}^2 = -4\lambda_0\lambda_4 + 2\lambda_1\lambda_3 + 4\lambda_0(4\wp_{31} - 3\wp_{22}) + 4\lambda_1\wp_{21} + 4\lambda_2\wp_{11}$.

where

$$\Delta_\wp = \wp_{32}\wp_{21} - \wp_{31}\wp_{22} + \wp_{31}^2 - \wp_{33}\wp_{11}.$$

5-18. Proposition. *For* $u = -2(\wp_{gg} - \lambda_{2g}/3)$ *and* $u(s, t_2, t_3) = u(\mathrm{t}_g, \dfrac{\mathrm{t}_{g-1}}{2^2}, \dfrac{\mathrm{t}_{g-2}}{2^4} + \dfrac{3}{2^4\lambda_{2g}}\mathrm{t}_{g-1})$ *obeys the first and the second KdV equations.*

Proof. Let us consider $g = 3$ case. If we regarded as $u = -2(\wp_{33} - \lambda_6/3)$, it is obvious that (1) in Example 3-14 becomes the KdV equation noting $\lambda_7 = 1$. By setting $2\partial_{\mathrm{t}_3} \times (2) + \partial_{\mathrm{t}_2} \times (1)$ and $\partial_{t_3} = 16\partial_{\mathrm{t}_1} + \dfrac{16\lambda_2}{3}\partial_{\mathrm{t}_2}$, we obtain the second KdV equation. From arguments of Baker [Ba1, Ba2], even for $g > 3$ the relations (1) and (2) maintain for g case. ∎

5-19. Remark.

(1) By above arguments, for given hyperelliptic curve $\overline{y}^2 = f(\overline{x})$, we can construct a solution of the first and second KdV equations. Further the compatibility of Lax system gives more general argument for the other equations in the KdV hierarchy. Then it implies that we explicitly showed the existence of an injective map

$$\mathfrak{M}_{\mathrm{hyp},g} \longrightarrow \mathfrak{M}_{\mathrm{KdV},g}.$$

This correspondence is valid even for degenerate curves.
s

(2) Our development of the quantized elastica after submitting this article is in [Mat7-10]. In [Mat7, 10], we showed more explicit function forms of quantized elastica over $\mathbb{C}$.

(3) After submitting this article, we knew the works of Buchstaber, Enolskii, Leykin and related people [reference in [BEL]] as an extension of parts of Baker's studies.

Now let us give another proof of Theorem 4-2 (2) and Proposition 4-33 (2).

5-20 Proposition. *Proposition 4-2 (2), 4-27 (3), and 4-33 (2) can be regarded as an approximation theory based upon the Weierstrass preparation theorem.*

Proof. Let us recall the moduli space of the KdV equations whose base ring is smooth functions and definition is in Proposition 4-32,

$$\mathbb{M}_{\mathrm{KdV}}{}^{\infty} = \{u \in \mathcal{C}^\infty(\mathcal{V}^\infty) \mid \partial_{t_n} u - \Omega_1^{n-1} u = 0 \text{ for } \forall n\ \}, \quad \mathcal{M}_{\mathrm{KdV}} = \mathbb{M}_{\mathrm{KdV}}/(t_1),$$

For an arbitrary $u \in \mathbb{M}_{\mathrm{KdV}}{}^{\infty}$, there is a unique spectrum $\mathrm{Spect}(L := -\partial_s^2 - u)$ up to its orbits, by solving the eigenvalue equation $(-\partial_s^2 - u)\psi_x = \overline{x}\psi_x$. We assume that the spectrum does not have finite gap $\{(c_1, c_2), (c_3, c_4), \cdots, (c_{2g-1}, c_{2g}), \cdots\}$ and then the corresponding characteristic equation becomes transcendental equation $\overline{y}^2 = f(x)$, where $f(x)$ is the transcendental function with zeros $(c_j)_{j=0,1,\ldots}$. Since u is a smooth function over S^1, $|u|$ is bounded the above. Hence around ∞ of the $\mathrm{Spect}(L)$, $L \sim -\partial_s^2$ and $\mathrm{Spect}(L)$ at ∞ is patched by the affine space $\mathbb{C}$; the width of gap converges to zero for $\overline{x} \to \infty$. Thus we can approximate $\mathrm{Spect}(L)$ by finite gap spectrum $\mathrm{Spect}(L_g) := \{(c_1, c_2), (c_3, c_4), \cdots, (c_{2g-1}, c_{2g}), (c_{2g+1}, \infty)\}$. The approximated potential u_g is given by the $\wp$ function of the hyperelliptic function $\overline{y}^2 = h_g(\overline{x}, 1)$ whose zero points are $(c_j)_{j=1,2,\cdots,2g+1}$. By using Weierstrass preparation theorem and taking appropriate g, we can approximate $f(\overline{x})$ by $h_g(\overline{x}, 1)$ for desired.

Hence up to the KdVH flow, u_g approaches to u for g approaches to ∞ from its construction. (For an arbitrary finite g, u_g is unique up to its orbits). Thus for an arbitrary u in $\mathbb{M}_{\mathrm{KdV}}{}^{\infty}$, there is a series of points u_g belonging to $F_g\mathbb{M}_{\mathrm{KdV}}$ such as $u_g \to u$ for $g \to \infty$ up to orbit. (We note that the finite type solutions does not depend upon the base rings $\mathcal{C}^\infty$ or formal power series.) Hence we have

$$\mathbb{M}_{\mathrm{KdV}}{}^{\infty} = \overline{\cup_g F_g \mathbb{M}_{\mathrm{KdV}}}.$$

Since $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ is a subset of $\mathbb{M}_{\mathrm{KdV}}{}^{\infty}$ and for an arbitrary curve $\gamma \in \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$, $u := \{\gamma, s\}_{\mathrm{SD}}$ has a unique value, the above statement is valid. ∎

5-21 Example. [E, T1, 2, Mat2] As an element γ in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ must satisfy $\gamma(s+L) = \gamma(s)$ in $\mathbb{P}$ and a reality condition $|\partial_s\gamma| = 1$. Even though the hyperelliptic function $\wp$ is a meromorphic function over $\mathcal{J}_g$, we can find a trajectory or real line in $\mathcal{J}_g$ which avoids the singularities and satisfies the reality and closed conditions. In other words, we will find $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ as a subset of $\mathbb{M}_{\mathrm{KdV}}$. We give examples of the $\gamma \in \mathbb{P}$ in terms of the local chart around the origin.

(1) genus $g = 0$ case: a circle with radius 1.

(2) genus $g = 1$ cases: $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}1}$ consists of two points:

2-1) Jacobi elliptic modulus $l = 1$ case

$$\gamma(s) = s - \frac{2}{\alpha}\left(\tanh(\alpha s) - \sqrt{-1}\sinh(\alpha s)\right).$$

2-2) Jacobi elliptic modulus $l = 0.908911\cdots$, which gives the eight shape loop in a complex plane $\mathbb{C}$ [E].

Here we note that in [Mum3], Mumford gave simple and deep expression of the shape of elastica, which shows the depth, importance and beauty of this problem. There he showed how the reality condition $|\partial_s \gamma| = 1$ restricts the moduli of elliptic curves.

§6. Cohomology of a Loop Space

As we mentioned in Introduction, in this section, we will digress from our analysis of the moduli of a quantized elastica and review arguments of a loop space over S^2 in the category of topological spaces **Top** whose morphism is a continuous map (isomorphism is homeomorphism, monomorphism is injective continuous map and so on). Studies on a loop space in **Top** are well-established and its cohomological properties are well-known as in the textbook of Bott and Tu [BT]. We can recognize the moduli space of a quantized elastica in $\mathbb{P}$ as a loop space in the category of the differential geometry **DGeom**. When we replace smooth functions with continuous functions and $\mathbb{P}$ with S^2 respectively, it is expected that the moduli space of a quantized elastica in $\mathbb{P}$ is related to that in **Top**. In this section, we will review a loop space in **Top** and show its cohomological properties.

6-1 Definition. [BT] E and X are topological space and X has a good cover $\mathfrak{U}$. A map $\pi : E \longrightarrow X$ is called a *fibering* if it satisfies the covering homotopy properties: for given a map $f : Y \longrightarrow E$ from an arbitrary topological space Y into E and homotopy $\overline{f}_t$ of $\overline{f} = \pi \circ f$ in X ($Y \times [0,1] \longrightarrow X$, $f_0 := f$), there is a homotopy f_t of f which covers $\overline{f}_t$; ($Y \times [0,1] \longrightarrow E$ such that $\overline{f}_t := \pi \circ f_t$).

6-2. Definition. [BT]

(1) The *path space* of X is defined to be the space $P(X)$ consisting of all the paths in X with initial point $*$:

$$P(X) := \{\text{maps } \mu : [0,1] \longrightarrow X \mid \mu(0) = * \in X \}.$$

(2) The *loop space* over X with a fixed point is defined by,

$$\Omega X = \{\mu : [0,1] \longrightarrow X |\ \mu(0) = \mu(1) = * \in X\}.$$

In the category of topological spaces **Top**, $\mathbb{P}$ and S^2 are identified by homeomorphism as its morphism. Thus we will give properties of the loop space over S^2 in **Top** as follows.

6-3. Theorem. [BT]

(1) *$P(S^2)$ is a fibering whose fiber is $\Omega(S^2)$:*

$$\begin{array}{ccc} \Omega(S^2) & \longrightarrow & P(S^2) \\ & & \downarrow \\ & & S^2 \end{array}$$

(2) *Its cohomology is torsionless and given by*

$$\mathrm{H}^q(\Omega S^2, \mathbb{Z}) = \mathbb{Z} \quad \textit{for } q \in \mathbb{Z}_{\geq 0}.$$

as a module and its algebraic properties are given by

$$\mathrm{H}^*(\Omega S^2, \mathbb{Z}) = \mathfrak{E}(x) \otimes_{\mathbb{Z}} \mathfrak{Z}_\gamma(e),$$

where x and e generators of $\mathrm{H}^1(\Omega S^2, \mathbb{Z})$ and $\mathrm{H}^2(\Omega S^2, \mathbb{Z})$ respectively ($\dim x = 1$ and $\dim e = 2$). Here $\mathfrak{E}(x)$ is the exterior algebra $\mathbb{Z}[x]/(x^2)$ and $\mathfrak{Z}_\gamma(e)$ is the divided polynomial algebra whose base is $(1, e, e^2/2, e^3/3!, \cdots)$. In other words, the generator of $\mathrm{H}^{2k+1}(\Omega S^2, \mathbb{Z})$ is $x \cdot e^k/k!$ and that of $\mathrm{H}^{2k}(\Omega S^2, \mathbb{Z})$ is $e^k/k!$.

In order to prove Theorem 6-3, we prepare two well-known results in algebraic topology and triangle category without proofs [BT].

6-4. Proposition. [BT] *For given a double complex $K = \oplus_{q,p\geq 0} K^{p,q}$, there is a spectral sequence $\{\mathrm{E}_r, d_r\}$ converging to the total cohomology $\mathrm{H}_D(K)$ such that each E_r has a bigrading with*

$$d_r : \mathrm{E}_r^{p,q} \longrightarrow \mathrm{E}_r^{p+r,q-r+1}$$

and

$$\mathrm{E}_1^{p,q} = \mathrm{H}_d^{p,q}(K), \quad \mathrm{E}_2^{p,q} = \mathrm{H}_\delta^{p,q}\mathrm{H}_d(K),$$

where d and δ are derivative: $d : K^{p,q} \longrightarrow K^{p+1,q}$ and $\delta : K^{p,q} \longrightarrow K^{p,q+1}$, $D = d + (-)^p\delta$.

We will consider the double complex for a fibering $\pi : E \longrightarrow M$,

$$K^{p,q} := C^p(\pi^{-1}\mathfrak{U}, \Omega^q).$$

Here $\mathfrak{U}$ is a ramification of M and Ω^q is a q-form along the fiber.

6-5. Proposition. (Leray-Hirsch theorem) [BT] *$\pi : E \longrightarrow X$ is a fibering with fiber F over simply connected topological space which has a good cover,*

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \mathrm{H}^q(F, A)),$$

where A is a commutative ring. If $\mathrm{H}^q(F, A)$ is a finitely generated A-module,

$$\mathrm{E}_2 := \mathrm{H}^*(X; A) \otimes \mathrm{H}^*(F; A).$$

Proof of Theorem 6-3. [BT] *Since $P(X)$ is contractive,*

$$\mathrm{H}^q(P(X)) = \begin{cases} \mathbb{Z} & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

and the spectral sequence must converge to $\mathrm{H}^p(P(X))$, E_2 must give isomorphism except 0-dimension.

$$\mathrm{E}_2 : \quad \begin{array}{c|ccccc} 5 & \vdots & \vdots & \vdots & \vdots & \ddots \\ 4 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots \\ 3 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots \\ 2 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots \\ 1 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots \\ 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots \\ \hline & 0 & 1 & 2 & 3 & \cdots \end{array}$$

Next we will consider the algebraic properties. From Proposition 6-5, E_2 is the tensor product $\mathrm{H}^*(\Omega S^2) \otimes \mathrm{H}^*(S^2)$. Let v be a two-form of S^2. Then if $\mathrm{H}^1(\Omega S^2)$ is denoted as $\mathbb{Z}x$, $\mathrm{E}_2^{0,1}$ is expressed by $\mathbb{Z}x \otimes 1$. The derivative d_2 in E_2, which is isomorphism, acts on $x \otimes 1$ as $d_2(x \otimes 1) = (1 \otimes v)$. Since $d_2(x^2 \otimes 1) = (d_2 x \otimes 1) \cdot x \otimes 1 - x \otimes 1 \cdot d_2 x \otimes 1 = (1 \otimes v)(x \otimes 1) - (x \otimes 1)(1 \otimes v) = 0$, we have $x^2 = 0$ because d_2 is isomorphism. Thus $d_2^{-1}(x \otimes v)$ is expressed by another generator e in $\mathrm{H}^2(\Omega S^2)$, which is algebraically independent of x. $d_2(e \otimes 1) = (x \otimes v)$. Since $d(ex \otimes 1) = e \otimes v$, ex is a generator in dimension 3. Similarly $d_2(e^2 \otimes 1) = 2ex \otimes v$ means that $e^2/2$ is a generator in dimension 4. In other words, we have a table such that,

$$\mathrm{E}_2 : \quad \begin{array}{c|ccccc} 5 & \vdots & \vdots & \vdots & \vdots & \ddots \\ 4 & e^2/2 \otimes 1 & 0 & 0 & 0 & \cdots \\ 3 & ex \otimes 1 & 0 & ex \otimes v & 0 & \cdots \\ 2 & e \otimes 1 & 0 & e \otimes v & 0 & \cdots \\ 1 & x \otimes 1 & 0 & x \otimes v & 0 & \cdots \\ 0 & 1 & 0 & 1 \otimes v & 0 & \cdots \\ \hline & 0 & 1 & 2 & 3 & \cdots \end{array}$$

Hence Theorem 6-3 is proved. ∎

6-6. Remark. [Br] Thought we showed the result on the loop space defined in Definition 6-2. However there are several studies on another loop space

$$\{\gamma : S^1 \hookrightarrow S^2 \mid \text{ smooth immersion}\},$$

and its cohomology, which differs from the result in Theorem 6-3 [Br]. This loop has a freedom of choice of starting points of S^1 in S^2. However in this article, we are concerned with a loop space with fixed point as we mentioned in Remark 2-15 and Definition 2-2. Accordingly we mentioned only the result.

§7. Topological Properties of Moduli $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$

As in previous section, we reviewed the cohomological properties of a loop space in **Top**, in this section we will argue its relation to our loop space in **DGeom** or the moduli space of a quantized elastica again. We believe that such considerations are important for the quantization of an elastica and the statistical mechanics of polymer physics [KL, Mat1, Mat2, Mat3].

The loop spaces in both **Top** and **DGeom** are infinite dimensional spaces when we regard them as manifolds in an appropriate sense. Even though it is not known that de Rham's theorem can be applicable to such an infinite dimensional manifold, it is expected that cohomological sequences should correspond to each other.

Precisely speaking, as we will show later, the closed condition and the reality condition $|\partial_s \gamma| = 1$ in the moduli space $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ makes its topological properties difficult. Thus we must tune the 0-dimension of the cohomology related to $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$. Then we will reach our second main Theorem 7-4, which implies that cohomology of $\mathbb{M}_{\mathrm{KdV}}$ reproduces Theorem 6-3 with $\mathbb{R}$ coefficients.

Since the loop space in **Top** is given with the fixed point, there is no translation freedom for the loop in S^2, which corresponds to our situation of quotient of $\mathrm{E}^0(\mathbb{C})$ in Definition 2-2. Further there is no freedom of change of the origin of the loop in **Top**. Hence we must compare ΩS^2 with $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ rather than $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$.

Further $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}0} \approx \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}1}/\mathfrak{W}^t{}_1 \approx \mathrm{pt}$, which should be regarded the same class because both these are zero dimension. The $\mathcal{F}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ might be natural sequence:

$$\mathcal{F}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} \ :\ \emptyset \to F_1\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} \hookrightarrow F_2\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} \hookrightarrow \cdots \hookrightarrow F_{g-1}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} \hookrightarrow F_g\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} \hookrightarrow F_{g+1}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} \hookrightarrow \cdots .$$

Noting $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g} := F_g\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}/F_{g+1}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$, as we are concerned only with its topological properties, let us consider the related complex of vector spaces,

$$\begin{aligned}\mathcal{G}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} \ :\ \emptyset \xrightarrow{\delta} F_1\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}/\mathfrak{W}^t{}_{0,1} \xrightarrow{\delta} \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}2}/\mathfrak{W}^t{}_2 \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g-1}/\mathfrak{W}^t{}_{g-1} \\ \xrightarrow{\delta} \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g}/\mathfrak{W}^t{}_g \xrightarrow{\delta} \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g+1}/\mathfrak{W}^t{}_{g+1} \xrightarrow{\delta} \cdots ,\end{aligned}$$

with trivial map $\delta = 0$ and $\delta^2 = 0$.

As each $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g}/\mathfrak{W}^t{}_g$ is a finite dimensional vector space $\mathbb{R}^{g-1}$ thanks to Proposition 4-30, we have de Rham complex $\mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g}$ $(g > 1)$,

$$\mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g} \ :\ 0 \to \Omega^0(\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g}/\mathfrak{W}^t{}_g) \xrightarrow{d} \Omega^1(\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g}/\mathfrak{W}^t{}_g) \xrightarrow{d} \Omega^2(\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g}/\mathfrak{W}^t{}_g) \xrightarrow{d} \cdots .$$

and $\mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}1}$

$$\mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}1} \ :\ 0 \to \Omega^0(F_1\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}/\mathfrak{W}^t{}_{0,1}) \xrightarrow{d} \Omega^1(F_1\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}/\mathfrak{W}^t{}_{0,1}) \xrightarrow{d} \Omega^2(F_1\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}/\mathfrak{W}^t{}_{0,1}) \xrightarrow{d} \cdots ,$$

where $\Omega^p(M)$ is the set of p-forms over M.

7-1. Proposition. *Let us consider a double complex $\mathcal{C}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$ with the derivative $D = d + (-)^g\delta$,*

$$0 \to \mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}1} \to \mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}2} \to \cdots \to \mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g-1} \to \mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g} \to \mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g+1} \to \cdots .$$

Then its cohomology,

$$\mathrm{H}^p(\mathcal{C}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}) := \oplus_g \mathrm{H}^{p-g+1}(\mathcal{D}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g}),$$

is given by $\mathrm{H}^0(\mathcal{C}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}) := \mathbb{R}$ and

$$\mathrm{H}^p(\mathcal{C}\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}) = \mathbb{R}dt_2 \wedge dt_3 \wedge \cdots \wedge dt_{p+1}, \quad p > 0.$$

Proof. First we note

$$\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}g}/\mathfrak{W}^t \approx \mathbb{R}^{g-1}, \text{ for } g \geq 1, \quad \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}1}/\mathfrak{W}^t \approx \mathrm{pt}.$$

Since we have for $n \geq 0$ [BT],

$$\mathrm{H}^p(\mathbb{R}^n) = \mathbb{R} \quad \text{for } p = 0.$$

Due to Poincaré duality, we have

$$\mathrm{H}^p(\mathbb{R}^n) = \mathrm{H}_c^{n-p}(\mathbb{R}^n),$$

if we write the compact support function valued cohomology by ${\mathrm{H}_c}^p$ [BT]. The generator is expressed by,

$$dt_2 \wedge dt_3 \wedge \cdots \wedge dt_g,$$

with a compact support function over there. ∎

First from Proposition 3-11 (5), let us interpret $\Omega : \partial_{t_n} \mapsto \partial_{t_{n+1}}$ as an endomorphism of tangent space of Jacobi varieties $T_* J_g$ of a hyperelliptic curve related to a point γ in $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$. Since the Jacobi variety is a quotient space of $\mathbb{C}^g$, its tangent space (and also its cotangent space) can be identified with $\mathbb{C}^g$: $T^* J_g \approx T_* J_g \approx \mathbb{C}^g$. Of course, we are concerned only with its real part $\mathbb{R}^g$. Then using the canonical duality in the real part $\mathbb{R}^g$,

$$< \partial_{t_n}, dt_m >= \delta_{n,m},$$

we can introduce an endomorphism Ω^{-1*} and Ω^* of $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$,

$$\Omega^* : dt_n \mapsto dt_{n-1} = \Omega^* dt_n, \quad \Omega^{-1*} : dt_n \mapsto dt_{n+1} = \Omega^{-1*} dt_n,$$

where $< \Omega\partial_{t_n}, dt_m >=< \partial_{t_n}, \Omega^* dt_m >$.

7-2. Definition. Let us define an endomorphism ϵ of $\mathbb{M}^{\mathbb{P}}_{\mathrm{elas}}$ by,

$$\epsilon := dt_2 \Omega^{-1*},$$

where Ω^{-1*} is regarded as a right action operator, $\epsilon^q = dt_2 \Omega^{-1*}(\wedge \epsilon^{q-1})$ for $q > 1$ and $\Omega^{-1*} \cdot 1 := 1$.

Then we have the properties of ϵ as follows.

7-3. Lemma.

(1) *We have the relation* $\epsilon^q \cdot 1 = dt_2 \wedge dt_3 \wedge \cdots \wedge dt_{q+1}$.

(2) ϵ *can be realized by* $\tilde{\epsilon}$,

$$\tilde{\epsilon} := \sigma \sum_{k=0} \epsilon_k, \quad \epsilon_0 := dt_2, \quad \epsilon_k := dt_{k+2} \wedge (dt_{k+1} i_{\partial_{t_{k+1}}}) \quad (k > 0),$$

where σ *is a permutation operator* $\begin{pmatrix} 1 & 2 & 3 & \cdots & q-1 & q \\ q & q-1 & q-2 & \cdots & 2 & 1 \end{pmatrix}$ *and* $i_{\partial_{t_k}}$ *is an inner product* $*$ *operator;* $i_{\partial_{t_k}} \cdot dt_l =< \partial_{t_k}, dt_l >= \delta^k_l$.

(3) *There is a ring isomorphism,*

$$\varphi_0 : \mathbb{R} \otimes_{\mathbb{R}} \mathfrak{E}(x) \otimes_{\mathbb{R}} \mathfrak{Z}_\gamma(e) \to \mathbb{R}[[\epsilon^2, dt_2]]$$

by

$$\varphi_0 : (e, x) \to (\epsilon^2, dt_2),$$

where the product in $\mathbb{R}[[\epsilon^2, dt_2]]$ *is defined by*

$$\epsilon * dt_2 = dt_2 * \epsilon := \epsilon \cdot dt_2, \quad \epsilon * \epsilon = \epsilon^2, \quad dt_2 * dt_2 = dt_2 \wedge dt_2 = s0.$$

Proof. (1): for example $\epsilon^2 \cdot 1 = dt_2 \Omega^{-1*}(\wedge dt_2 \Omega^{-1*}) \cdot 1 = dt_2 \wedge dt_3$ and this can be extended to general case. (2): noting $\epsilon_k^2 = 0$, $(k \geq 0)$, straightforward computations gives the results. (3): noting Theorem 6-3, it is obvious.∎

Here we will note that $\epsilon : \mathrm{H}_c^g(\mathbb{R}^g) \to \mathrm{H}_c^g(\mathbb{R}^{g+1})$ generates the sequence $\mathbb{R}^g \hookrightarrow \mathbb{R}^{g+1}$, and thus ϵ^m could be regarded as a generator of the filter topology of $\mathbb{M}_{\mathrm{elas}}^{\mathbb{P}}$ and $\mathbb{M}_{\mathrm{KdV}}$. Thus it means that we can evaluate the moduli space of a quantized elastica $\mathbb{M}_{\mathrm{elas}}^{\mathbb{P}}$ using the induced topology and ϵ as in Proposition 4-26.

Finally we reach our third main theorem.

7-4. Theorem. *By setting $e = \epsilon^2$, $x = dt_2$, the cohomology $\mathrm{H}^q(\mathcal{CM}_{\mathrm{elas}}^{\mathbb{P}})$, is a ring isomorphic to $\mathrm{H}^q(\Omega S^2, \mathbb{R})$,*

$$\phi : \mathrm{H}^*(\mathcal{CM}_{\mathrm{elas}}^{\mathbb{P}}) \xrightarrow{\sim} \mathrm{H}^*(\Omega S^2, \mathbb{R}).$$

7-5. Remark.

(1) The closed condition $\gamma(s+L) = \gamma(s)$ for some L and the reality condition $|\partial_s \gamma| = 1$ are too strong. For example due to the condition, $\mathcal{M}_{\mathrm{elas}0}^{\mathbb{P}}$ and $\mathcal{M}_{\mathrm{elas}1}^{\mathbb{P}}$ consist only of disjoint points as mentioned in Examples 5-21 and [Mat2]. Thus if we assign real vector bases each point, these cohomology might be $\mathrm{H}^p(\mathcal{M}_{\mathrm{elas}0}^{\mathbb{P}}) = \mathbb{R}\delta_{0,p}$ and $\mathrm{H}^p(\mathcal{M}_{\mathrm{elas}1}^{\mathbb{P}}) = \mathbb{R} \oplus \mathbb{R}\delta_{0,p}$. These phenomena come from a "elasticity" in the category **DGeom** but we wish to consider the topological properties of the loop space in **DGeom**. Thus we have replaced $\mathcal{M}_{\mathrm{elas}}^{\mathbb{P}}$ with $\mathcal{CM}_{\mathrm{elas}}^{\mathbb{P}}$ by loosing strongness of the condition and make its topology weak; it implies a replacement to fewer open sets. This replacement comes from modulo computations in the gauge transformation by $\mathfrak{W}^t$ in the KdV equations, using the natural immersion $i_{\mathrm{KdV}} : \mathbb{M}_{\mathrm{elas}}^{\mathbb{P}} \to \mathbb{M}_{\mathrm{KdV}}$.

However for a sufficiently large g case, the closed condition and the reality condition might not have serious effects. Then the quotient by gauge transformation can be also guaranteed by the fact that each moduli space of compact Riemannian surface of genus g is simply connected [McLac]. Accordingly we consider that the replacement is not worse.

The isomorphism ϕ could be regarded as a functor between triangle categories of loop spaces in **DGeom** and **Top** and a quasi-isomorphism between $\mathcal{CM}_{\mathrm{elas}}^{\mathbb{P}}$ and ΩS^2 [Br]. (These objects in the triangle categories are vector spaces given by ϵ^n and (x^a, e^m) respectively. The morphisms are multiplications as their ring structures.)

(2) From the definition, ϵ^m can be regarded as a map from $\mathrm{H}^q(\mathcal{CM}_{\mathrm{elas}}^{\mathbb{P}})$ to $\mathrm{H}^{q+m}(\mathcal{CM}_{\mathrm{elas}}^{\mathbb{P}})$. We should regard that this map comes from the properties of vertex operator, which change the genus of curves [DJKM] and $\epsilon^m \cdot 1$ is interpreted as a topological base of $\mathcal{CM}_{\mathrm{elas}}^{\mathbb{P}}$.

(3) The operator ϵ induces the complexes $\mathcal{FM}_{\mathrm{elas}}^{\mathbb{P}}$ and $\mathcal{GM}_{\mathrm{elas}}^{\mathbb{P}}$. This essentially exhibits the topology of Sato theory because in Sato theory [S, SN, SS], the existence of the gauge transformation $\mathfrak{W}^t$ is a key factor. Theorem 7-4 means that its topology is as strong as that of a loop space in **Top**. It implies that the topology of Sato theory is too weak to lead us to express fine structure of the moduli space as Harris and Morrison pointed out in [HM, p.44-5]; they stated that the geometrical approach in [Mul] does not influence the study of the moduli space of algebraic curves including $\mathfrak{M}_{\mathrm{hyp}}$ [HM]. In fact as mentioned in [HM, Mum0], the moduli space of $\mathfrak{M}_{\mathrm{hyp}}$ is, in general, very complicate but our approach is not so difficult. Accordingly we wish to obtain stronger topology to express the moduli space. We hope that the day comes

that the studies on quantized elastica are connected with those of $\mathfrak{M}_{\rm hyp}$ as Euler did for the case of genus one [E, T1, 2].

(4) As we will comment in Remark 8-9, the correspondence between loop spaces in **DGeom** and **Top** can be extended to higher dimensional loop spaces by considering recent result of a quantized elastica in $\mathbb{R}^3$ [Mat4].

§8. Discussion

8-1. Although we have correspondence between homological properties of ΩS^2 in **Top** and those of $\mathcal{M}^{\mathbb{P}}_{\text{elas},g}$ in **DGeom**, there is open problem for a correspondence of homotopy group between them, *e.g.*,

$$\pi_{q-1}(\Omega S^2) = \pi_q(S^2) \quad (q \geq 2),$$

$$\pi_{q-1}(\Omega S^2) \times \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{for } q = 1, 2 \\ 0 & \text{otherwise} \end{cases} .$$

8-2. [Mat2] We will consider $\gamma \in \mathbb{M}^{\mathbb{C}}_{\text{elas}}$ in this remark. By defining

$$v = \left(\frac{\partial_s^2 \gamma}{2\sqrt{-1}\partial_s \gamma} \right),$$

this problem is related to the quantization of an elastica in $\mathbb{C}$,

$$Z[\beta] = \int_{\mathcal{M}^{\mathbb{C}}_{\text{elas}}} D\gamma \exp(-\beta \int_{S^1} v^2 \mathrm{d}s).$$

For $\beta > 0$, the domain of $E = \int_{S^1} v^2$ can be extended to ∞-point and we will define

$$\overline{\mathcal{M}^{\mathbb{C}}_{\text{elas}}} = \{\gamma : S^1 \longrightarrow \mathbb{C} \mid \gamma \text{ is continuous}, \quad |\partial_s \gamma(s)| = 1\} / \sim .$$

In other words, as we assign the energy of γ with wild shape to ∞-point of E, it does not contribute the partition function Z. Then we can regard the partition function as

$$Z : \overline{\mathcal{M}^{\mathbb{C}}_{\text{elas}}} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}.$$

The integral region in Z is recognized as $\overline{\mathcal{M}^{\mathbb{C}}_{\text{elas}}}$. Due to our Theorem 3-4, we have a natural projection operator Π_E:

$$\Pi_E : \mathcal{M}^{\mathbb{C}}_{\text{elas}} \longrightarrow \mathcal{M}^{\mathbb{C}}_{\text{elas},E}, \quad \Pi_E^2 = \Pi_E.$$

We have a spectral decomposition,

$$1_{\mathcal{M}^{\mathbb{C}}_{\text{elas}}} = \int dE \Pi_E.$$

Hence the partition function becomes

$$Z[\beta] = \int dE \ \mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\text{elas},E}) e^{-\beta E},$$

where $\mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\text{elas},E})$ means the volume of $(\mathcal{M}^{\mathbb{C}}_{\text{elas},E})$.

Here we will comment on a question why we can use the concept of the orbits of "kinematic" system even though in the noncommutative algebra, one sometimes encounters nonsense of concept of orbit, *e.g.*, Kronecker foliation [C]. Even in quantized problem, we can go on to use the concept of orbit and commutative geometry even though the dimension of the orbit space need not be finite.

Let extend to the domain of $\beta \in \mathbb{R}_{\ge 0}$ to $\mathbb{R}_{\ge 0} + \infty$. Note that as the inverse image,

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas,cls}} = Z^{-1}(Z(\infty)),$$

the classical moduli space of the harmonic map of the elastica depending upon the boundary condition is naturally immersed in our moduli space $\overline{\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}}$. In other words, our analysis naturally contains Euler's perspective of the classical elastica [E, T1, 2, L].

8-3. Due to the projection operator, we can define the order in the moduli space $\mathcal{M}^{\mathbb{P}}_{\mathrm{elas}}$. Noting that the energy E is real in $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$, let

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},<E} := \coprod_{E'<E} \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E'}.$$

For $E_1 < E_2$, we have

$$\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},<E_1} \subset \mathcal{M}^{\mathbb{C}}_{\mathrm{elas},<E_2}.$$

Then the moduli space $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas}}$ is an ordered space.

8-4. The operator ϵ in Lemma 7-3 can be regarded as a creation operator in the quantum field theory. The vacuum state is regarded as 1. We can define the dual space of $\mathcal{V}^\infty$; $< e^m, e_n >= \delta^m_n$ where $e_n = dt_n$ and $e^m = \partial_{t_m}$.

Further by noting ϵ modulo ϵ^2, we can reconstruct $\mathcal{CM}^{\mathbb{P}}_{\mathrm{elas}}$ in Lemma 4-31. On the other hand, we can introduce the micro-differential operator e^m $(m\mathbb{Z})$ as the base of $\mathcal{CM}^{\mathbb{P}}_{\mathrm{elas}}$ as in the Definition 3-1 and Proposition 7-1. Then as the dual of $\mathcal{CM}^{\mathbb{P}}_{\mathrm{elas}}$, we can define e_m $(m < 0)$ and the vacuum of this field operator in the quantum field theory has affine structure as physicists think.

8-5. In the differential operator ring, $\mathfrak{D}^s$, the integral $\int_{S^1} \partial_s u = 0$ means that since the integral is linear map, its kernel belongs to $\mathfrak{D}^s/\partial_s\mathfrak{D}^s$.

Using the Definition 3-7 and Proposition 3-11, let us define,

$$h := \sum_j h_j dt^j, \quad \delta := \sum_j dt^j \partial_{t_j}, \quad \mathfrak{a} = u ds.$$

we have the transformation in $(\mathfrak{D}^s/\partial_s\mathfrak{D}^s)$:

$$\delta\mathfrak{a} = \tilde{\Omega} h, \quad \tilde{\Omega} := ds\partial_s \frac{\delta}{\delta u},$$

$$\delta * h = 0.$$

This relation is called Becchi-Rouet-Stora (BRS) relation [LO, Mat2].

8-6. We will introduce a dilatation flow

$$\partial_t \psi_x = t\partial_s \psi_x.$$

The intersection between this flow and the KdV flow is governed by the Painlevé equation of the first kind,

$$s = 3u^2 + \partial_s^2 u.$$

This statement can be proved as follows. Since the KdV flow in Remark 3-8 is given by $B_1 = u$ while this flow $B_1 = t$. Hence $u = t$ and the KdV flow becomes

$$\partial_t u = 1 = \partial_s(3u^2 + \partial_s^2 u),$$

and we obtain the Painlevé equation of the first kind [In, Mat2].

8-7. Since the Schwarz derivative u is invariant for $\mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$ transitively acts upon $\mathbb{P}$, we can regard $\mathcal{M}_{\mathrm{elas}}^{\mathbb{P}}$ as

$$\Omega\mathrm{SL}_2(\mathbb{C}) := \{\gamma : S^1 \hookrightarrow \mathrm{PSL}_2(\mathbb{C}) \mid \gamma(0) = 1\}.$$

Because $\gamma(s) = g_s\gamma(0)$ for $g \in \mathrm{PSL}_2(\mathbb{C})$, we have the condition $g(0) = g(2\pi)$. As Witten pointed out, for a loop space we can naturally construct its tangent space as a loop space of the tangent space of the target space [Wi]. In other words, we can naturally define a loop algebra $\Omega\mathrm{sl}_2(\mathbb{C})$. In the loop algebra, we have only the condition $g^{-1}dg(0) = g^{-1}dg(2\pi)$ using $g \in \mathrm{SL}_2(\mathbb{C})$, which is not stronger condition than the condition $g(0) = g(2\pi)$. Since there is a smooth map from S^1 to S^1 as $\mathrm{Diff}(S^1)$, we obtain an expression of the loop algebra,

$$\mathrm{Diff}(S^1) \otimes \mathrm{sl}_2(\mathbb{C}) \oplus \mathbb{C},$$

which acts upon ${\mathbb{M}_{\mathrm{KdV}}}^{\infty}$ in Proposition 4-32 with the weaker condition.

In fact, the KdV flow has bi-hamiltonian structure and 2-cocycle

$$\omega_{\underline{\Omega}}(X, Y) := \omega(\underline{\Omega} X, Y) + \omega(X, \underline{\Omega} Y).$$

Using ordinary functional derivative (Gatuex derivative $\delta u(y)/\delta u(x) = \delta(x - y)$), we can write down the (second) Poisson relation,

$$\{u(s), u(s')\} = \underline{\Omega}\delta(s - s'),$$

where $\delta(s)$ is the Dirac δ-function.

Let

$$l_n := \frac{1}{2\pi}\int \mathrm{d}s\ u_\kappa e^{\mathrm{i}sn}.$$

denote its Fourier component. Then it obeys the semi-classical Virasoro algebra,

$$\{l_n, l_m\} = (n - m)l_{n+m} + n(n^2 - 1)\delta_{n+m,0}.$$

where the second term the unit central charge. We have the Virasoro algebra.

Using the topological relation $\mathbb{C}^* \sim S^1$, the problem of conformal field theory is reduced to that of the loop algebra. Thus our relation can be also interpreted in the regime of the conformal field theory. Thus it is clear that our problem is related to the two dimensional quantum gravity [HM].

8-8. It is known that for $H_0 := \int u ds$, the second Poisson structure of H_0 reproduces the KdV equation; when the second Poisson bracket is defined as

$$\{X, Y\}_{\underline{\Omega}} = \omega_{\underline{\Omega}}(X, Y),$$

$\partial_t u = \{u, H_0\}_{\underline{\Omega}}$ is $\partial_t u + 6u\partial_s u + \partial_s^3 u = 0$.

If we will used the Hamiltonian H_n of the higher dimensional KdV as the energy functional of the system, we will have another decomposition,

$$\mathcal{M}^{\mathbb{P},(n)}_{\mathrm{elas}} = \coprod \mathcal{M}^{\mathbb{P},(n)}_{\mathrm{elas},E}$$

$$\mathcal{M}^{\mathbb{P},(n)}_{\mathrm{elas},E} := \left\{ \gamma_t \in \mathcal{M}^{\mathbb{P}}_{\mathrm{elas}} | \ H_n - E = 0 \ \right\}.$$

The space is determined by the $n(>1)$-th KdV hierarchy,

8-9. [BT, Mat4] According to the results in [BT], we have the relation

$$\mathrm{H}^q(\Omega S^n, \mathbb{Z}) = \mathbb{Z} \quad \text{ for } q = 0 \ \text{ modulo } n - 1.$$

As we mentioned in Remark 7-5, it is expected that the moduli space of a quantized elastica in S^n has similar cohomological properties. In fact, one of these authors calculated the quantized elastica in $\mathbb{R}^n$ and obtained the same structure of the moduli space of a quantized elastica in $\mathbb{R}^n$ [Mat4].

8-10. We wish to know the volume of each $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E}$. However this problem is not easy. In fact as pointed out in [HM], the soliton theory might not affect to get any information of the structure of $\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E}$.

In other words, our Theorem 7-4 means that the filter topology in the soliton theory is too week and is equivalent with the topological properties of the loop space. It might have no effect on the study of geometrical future of moduli space of hyperelliptic curve. Thus we believe that we must go beyond the ordinary soliton theories to another theoretical world for the study of moduli space of a quantized elastica as Euler investigated the elliptic functions by studying the shape of classical elasticas [E, T1, 2 ,L, We].

8-11. First we will note the relations for $\mathbb{P}$, $\mathbb{C}$ and upper half complex plane $\mathbb{H}$;

$$\begin{array}{ccccc} \mathbb{P} & : & \mathrm{PSL}_2(\mathbb{C}) & : & \dfrac{a\gamma + b}{c\gamma + d} \\ \mathbb{C} & : & & : & a\gamma + b \\ \mathbb{H} & : & \mathrm{PSL}_2(\mathbb{R}) & : & \dfrac{a\gamma + b}{c\gamma + d} \end{array}$$

We showed that loops on $\mathbb{P}$ are related to the KdV flow and that loops on $\mathbb{C}$ are related to the MKdV flow. Next we should consider loops on $\mathbb{H}$.

8-12. One of solutions of

$$(-\partial_s^2 - \frac{1}{2}\{\gamma, s\}_{\mathrm{SD}})\psi = 0$$

is given by $1/\sqrt{\partial_s\gamma}$. The coordinate transformation for the Diff(S^1) leads us to redefine ψ as the invariant form $\sqrt{ds/d\gamma}$. This reminds us of the prime form and the Dirac field which has a half weight as same as the theta function [Mum2, Mat7, 9].

In fact, for a curve in $\mathbb{C} \subset \mathbb{P}$, there is a natural topology of γ induced form the distance in $\mathbb{C}$, which is given by the Frenet-Serret relation:

$$\begin{pmatrix} \partial_s & k/2 \\ -k/2 & \partial_s \end{pmatrix} \begin{pmatrix} 1/\sqrt{\partial_s\gamma} \\ i/\sqrt{\partial_s\gamma} \end{pmatrix} = 0.$$

This operator is regarded as the Dirac operator. The Dirac operator could be regarded as a translator from the category of analysis to the category of geometry. Hence as we are dealing with the topology of the Dirac operator, we might have a stronger topology of the curve.

We can extend this structure to a conformal surface in $\mathbb{R}^3$ as the generalized Weierstrass relation [Kno, KL, Mat5, 6].

We note that this Dirac operator (and the Schrödinger operator in Proposition 2-8) defined upon the loop space differs from the Dirac operator of Witten in [Wi] because Witten's one is related to the conformal field theory and the ordinary string which is determined by intrinsic properties whereas ours are related to the extrinsic Polyakov string [KL, Mat5, 6].

8-13. As we noticed in 8-2, the partition function Z can be expressed by

$$Z = \int dE \mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E}) e^{-\beta E},$$

where $\mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E})$ is formally represented by

$$\mathrm{Vol}(\mathcal{M}^{\mathbb{C}}_{\mathrm{elas},E}) = \sum_g \int_{\mathfrak{M}^{\mathbb{C}}_{\mathrm{elas},E,g}} d\mathrm{vol}(J) \int_J dt_2 dt_3 \cdots dt_g,$$

where $d\mathrm{vol}(J)$ is the volume form around a point J in $\mathfrak{M}^{\mathbb{C}}_{\mathrm{elas},E,g}$ and $\mathfrak{M}^{\mathbb{C}}_{\mathrm{elas},E,g} := \mathfrak{M}^{\mathbb{C}}_{\mathrm{elas},E} \cap \mathfrak{M}^{\mathbb{C}}_{\mathrm{elas},g}$. Then we will leave integral over t_2, in the above expression and obtain the time t_2 depending partition function,

$$Z[t_2] = \int dE \sum_g \int_{\mathfrak{M}^{\mathbb{C}}_{\mathrm{elas},E,g}} d\mathrm{vol}(J) \int_J dt_3 \cdots dt_g e^{-\beta E}.$$

Similarly we obtain $Z[t_2, t_3, \cdots, t_g]$, which is a generating function [R]. Then we can expect that it might obey the KdV equation or related equation. This situation might be related to with Witten's conjecture and Kontsevich's theorem [HM].

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